Hyperbolic Geometry

The fact that an essay on geometry such as this must include an additional qualifier signifying what kind of geometry is to be discussed is a relatively new requirement. From around 300 B.C. until the early 19th century, 'geometry' meant Euclidean geometry, for there were no competing systems to rival the intrinsic truth of Euclid's geometry put forth in his *Elements*. But a particularly troublesome piece of the Euclidean puzzle began to lead thinkers down new avenues of geometrical description. That troublesome piece of Euclidean geometry was the now infamous fifth axiom, commonly referred to as the parallel postulate. This axiom was by far the most awkward of Euclid's foundational axioms, and Euclid himself was reluctant to include it among the simpler, more intuitive first four. Moreover, Euclid was able to derive a very extensive body of propositions (the first 28) just using the first four axioms, making the clumsy fifth stand out of place even more.

Many attempts to derive the parallel postulate from the original four axioms were made by Euclid and countless mathematicians for the next two thousand years to no avail. Gauss is said to be one of the founders of non-Euclidean geometry because he took a different attitude towards the parallel-postulate problem. After many years of trying unsuccessfully to derive the parallel postulate, Gauss was the first mathematician to abandon the prejudice that the parallel postulate was necessary to ensure a consistent geometrical system. That is, he set out to develop a geometry which denied the fifth postulate without the expectation that a contradiction would surface (Coxeter 7). And lo and behold, several different geometries denying the parallel postulate were worked out and shown to be internally consistent. The two most well-known of which are hyperbolic and elliptic geometry. It is generally recognized nowadays that the inception of non-Euclidean geometry occurred almost simultaneously and independently by the hands of three separate mathematicians, Gauss, Bolyai, and Lobachevsky. However it is also very common when one must single out an individual founder to give this distinction to Lobachevsky as he worked out the implications farther than Bolyai and had the courage to publish his findings, which Gauss did not.

Euclid's five basic Postulates as stated in his *Elements* were approximately as follows:

I.) A straight line may be drawn from any one point to any other point.

II.) A finite straight line may be extended to any length in a straight line.

III.) A circle may be described with any center at any distance from that center.

IV.) All right angles are equal.

V.) If a straight line meets two other straight lines, so as to make the two interior angles on one side of it together less than two right angles, the other straight lines will meet if produced on that side on which the angles are less than two right angles.

Parallel lines are then defined as lines that do not meet no matter how far they are extended in either direction. Obviously the fifth postulate sticks out like a sore thumb as compared to the more self-evident I-IV. Becuase of its complicated appearance, many thought that it could be deduced as a proposition from the first four and Euclid's other "common assumptions" concerning magnitude. However the best anyone could do was to come up with equivalent assumptions to the fifth postulate such as the following:

1.) Two parallel lines are equidistant throughout their entire length.

2.) If a line intersects one of two parallels, it also intersects the other.

3.) Given a triangle, we can construct a similar triangle of greater or lesser side-length.

4.) The sum of the angles of a triangle is equal to two right angles.

5.) Three non-collinear points always lie on a circle.

A different phrasing of (V) which we have been referring to as the parallel-postulate states:

V'.) *Given a line and a point not on it, there is exactly one line going through the given point that is parallel to the given line.*

This is also known as Playfair's axiom and is commonly used in the place of Euclid's Fifth postulate because of its relative succinctness. To retain all the properties of Euclidean geometry the parallel postulate or an equivalent assumption such as listed above must be included.

While maintaining the first four postulates let's consider what happens if we make an additional assumption that is not equivalent to the parallel postulate but rather contradicts it. What is commonly known as hyperbolic geometry replaces Euclid's fifth axiom with the following:

H-V'.) *Given a line and a point not on it, there is more than one line going through the given point that is parallel to the given line*.

Those assumptions 1-5 above which were equivalent with the Euclidean parallel postulate are transformed when we adopt the hyperbolic parallel postulate to the following:

1'.) Two lines cannot be indefinitely equidistant.

2'.) A line may intersect one of two parallels without intersecting the other.

3'.) Similar triangles are necessarily congruent.

4'.) The sum of the angles of a triangle is less than two right angles.

5'.) Three points may be neither collinear nor concyclic.

These statements certainly appear curious at first glance to the student brought up in traditional Euclidean geometry. All similar triangles *must* be of the same size? Lines cannot be indefinitely equidistant?

These propositions seem to go against our common sense. However we must remember that much of what we consider "common sense" is a construct of our social education. It is not wrong to teach Euclidean geometry in this way, it is the most useful geometry for the practical world and children often need what they're learning to have the stamp of ultimate truth given by authority. However as critical thinkers we must deconstruct those social prejudices when looking at notions which might challenge our common sense. The detailed critical thinking that went into constructing non-Euclidean geometries in fact led to the revolution of in depth ontological justification in all mathematics and sciences which has led to the present freedom and success modern science enjoys.

With this in mind, let us now carefully examine the elementary facets of Hyperbolic geometry and see how we are led to such fascinating new geometric properties. The reader should keep in mind that only the propositions of Euclid which require the fifth postulate are invalid for our use in Hyperbolic geometry. Thus the first 28 propositions may be employed as they can be derived from just the first four postulates. The reader may refer to these at their leisure in the appendix at the end of this paper. Certain fundamental axioms of order, continuity, and congruence will also be assumed as valid for our discourse. One such axiom of order which will come up more than once is the Axiom of Pasch. This states that *if a straight line cuts one side of a triangle and does not pass through a vertex, it will also cut one of the other sides (if it does pass through a vertex it cuts the opposite side)*. The Postulate of Dedekind is a principle of continuity which states that: *If all points of a straight line fall into two classes, such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes.* These axioms and several of Euclid's first 28 propositions will be utilized in the proofs. The reader should also note that angles in hyperbolic geometry behave as they do in standard Euclidean geometry. The four angles created by two intersecting lines always add up to a constant, 2π, while the angle measure of a straight line is π. Two lines forming an angle measuring $\pi/2$ are said to perpendicular.

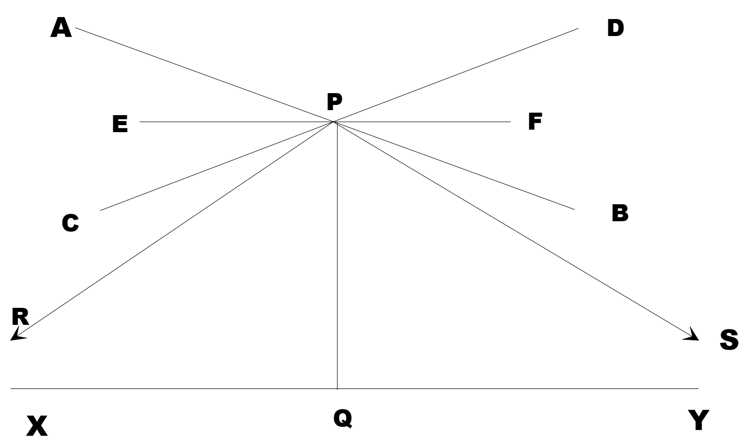
The terms equidistant and parallel are not synonymous, though in Euclidean geometry they do coincide. This connection has given rise to much confusion as to what the term parallel really indicates. For the purposes of our discourse, we shall maintain the following definition for parallelism. AA' is said to be parallel to BB' if and only if:

1.) AA' does not meet BB', both being produced indefinitely, and

2.) every ray drawn through A within the angle BAA' meets the ray BB'.

With the Hyperbolic parallel postulate **(H-V')** in mind let us put forth the first fundamental theorem of hyperbolic geometry.

Theorem 1: *If ℓ is any line and P is any point not on ℓ, then there are always two distinct lines, PS and PR, through P which do not intersect ℓ, and which are such that every line through P lying within the angle containing the perpendicular PQ intersects ℓ, while every other line through P does not.*



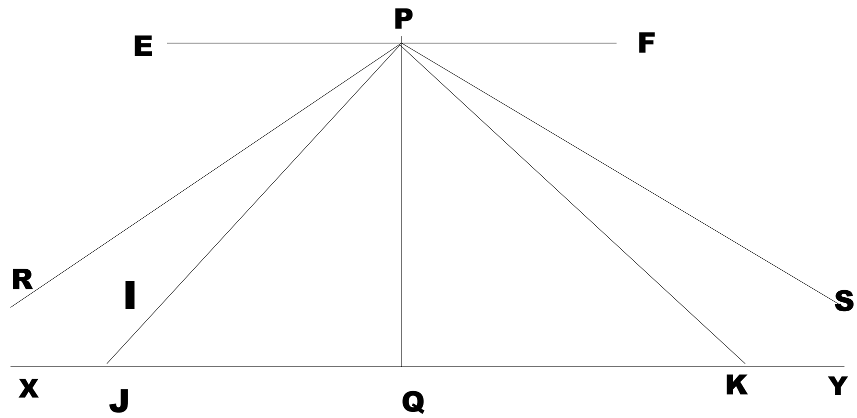
Proof: Let AB and CD be two lines through P which do not intersect ℓ. No line such as EF lying within the angles APC and DPB, which do not contain the perpendicular from P to ℓ, PQ, will intersect ℓ. This follows because if EF were to intersect ℓ when extended to the right, then since

PF and PQ both intersect ℓ, PB would also intersect ℓ by the Axiom of Pasch, yielding a contradiction with our hypothesis.

If one were to rotate the perpendicular PQ counterclockwise one notices that for a certain amount of rotation it will continue to intersect ℓ, and then eventually it will cease to intersect. Thus there two sets of lines through P, those that intersect ℓ and those that do not. A basic postulate of continuity called the Postulate of Dedekind provides that there exists a certain line through P where this division occurs. Thus this line must be either the last line which intersects ℓ or the first line which does not. However there is not a last line that intersects ℓ since wherever a given line intersects ℓ one only needs to measure off a little more distance on ℓ and get another intersecting line rotated further clockwise. Thus this dividing line must be the first line which does not intersect ℓ. A similar situation unfolds if we rotate PQ clockwise. Thus there exists two lines PR and PS through P which do not intersect ℓ, and which are such that every line lying within angle RPS does intersect ℓ.   (Wolfe 66)

Definition: The lines PR and PS described above are the lines through P **parallel** to ℓ. PR is sometimes referred to as the left-hand parallel while PS is referred to as the right-hand parallel to distinguish the two. We can also say that directed line PS is parallel to directed line XY, while directed line PR is parallel to directed line YX. When we say that a line is parallel to another line in a certain direction this indicates that whatever direction indicated is the side where the perpendicular between the two lines is getting smaller. The direction opposite to that of parallelism is the direction where the perpendicular between the two lines is going getting larger.

Theorem 2: *The angles RPQ and SPQ, formed by the perpendicular from P to ℓ and each parallel line are equal and acute.*



Proof: Suppose they are not equal, for example ∠RPQ > ∠SPQ. Measure angle IPQ as equal to angle SPQ on the same side of PQ as PR. Since it lies within angle RPQ, PI will cut ℓ at point J. Now on the opposite side of PQ measure off QK equal to QJ on ℓ. Draw PK. Since triangles QPK and QPJ are right triangles with two equal sides, they are congruent (Euclid, prop. 4) and ∠QPK = ∠QPJ = ∠QPS. However PS does not intersect ℓ as PK does since PS is a parallel to ℓ, thus we have reached a contradiction. Thus one concludes angles RPQ and SPQ are equal.

Now to show angles RPQ and SPQ are acute we again proceed by contradiction. Suppose they are right angles, then by the definition of right angles their sum would be π and PR and PS would lie in a straight line through P perpendicular to PQ. However it has already been stated that there are two distinct lines parallel to ℓ through P. Any such line which does not intersect ℓ has a ray which lies within angle RPS, thereby violating **theorem 1**. Suppose RPQ and SPQ are obtuse. If this were so then line EF, perpendicular to PQ through P, would lie within angle RPS and would have to intersect ℓ by **theorem 1**. However a line cannot intersect a line it shares a perpendicular with by proposition 28 of Euclid, thus ∠RPQ and ∠SPQ must be acute angles. (Wolfe 67)

Definition: The angle described above formed by the perpendicular from P to ℓ and a line through P parallel to ℓ is called the **angle of parallelism**. This angle is denoted by $\Pi(h)$ where h is the length of the perpendicular. It will be shown later on that the angle of parallelism depends solely upon the length of h.

Parallel lines in Hyperbolic geometry share some of the same properties as their Euclidean counterparts. Three such properties follow:

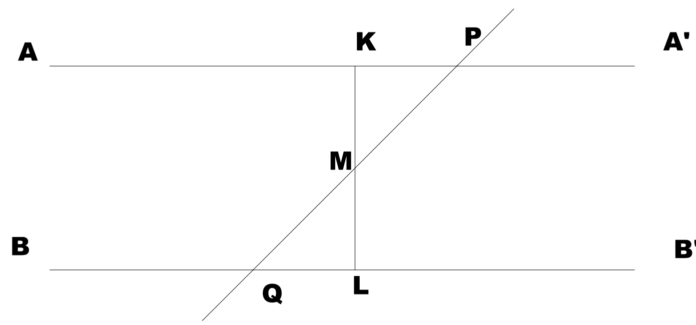
1.) *Property of Transmissibility:* If a straight line is the parallel through a given point in a certain direction to a given line, it is, at each and every one of its points, the parallel in that direction to the given line. In other words, the property of parallelism is maintained, in a given direction, throughout the whole length of the line.
2.) *Property of Reciprocity:* If one line is parallel to a second, then the second is parallel to the first. If AA' || BB' then BB' || AA'.
3.) *Property of Transitivity:* If two lines are both parallel to a third line in the same direction, then they are parallel to one another. (Sommerville 32)

These similarities do not mean that parallels in Hyperbolic and Euclidean geometry behave in total accordance. Quite the contrary is the case. As we shall see, the next property of Hyperbolic parallels sharply distinguishes them from the familiar Euclidean parallel.

The distance between two parallels diminishes in the direction of parallelism and tends to zero; in the other direction the distance increases and tends to infinity.

In proving this we will develop the main terms, properties and relations that characterize hyperbolic geometry. It is a rather complicated proof so first we must prove some more basic theorems, and we will also encounter some of the most interesting and astounding results of

Theorem 3: *If a transversal meets two lines making the sum of the interior angles on the same side equal to two right angles, the two lines cannot meet and are not parallel.*

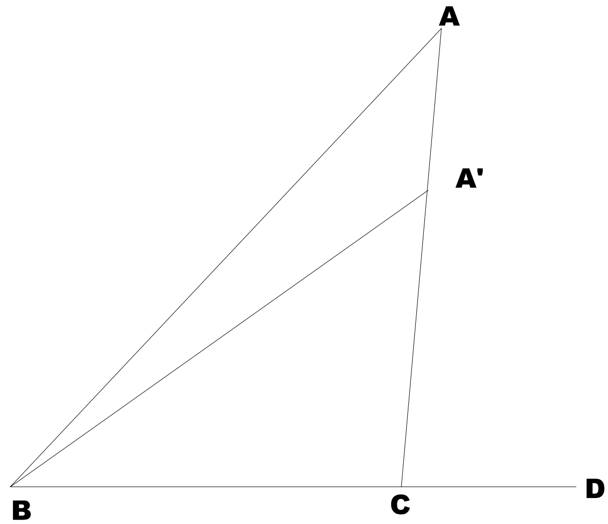


Proof: Let PQ be a transversal cutting two lines AA' and BB' at points P and Q, respectively, such that ∠APQ + ∠PQB equals two right angles. Since they constitute a straight line, ∠PQB + B'QP = π, and therefore alternate angles ∠APQ and ∠B'QP are equal.

Bisect PQ at M and draw MK ⊥ AA', ML ⊥ BB'. The triangles MKP and MLQ are congruent and ∠KMP = ∠LMQ. Thus KML is a straight line perpendicular to both AA' and BB'. Since ∠PKM and ∠QLM are equal by the congruence of triangles, if AA' meets BB' on one side, they must meet on the other. It is not possible in hyperbolic geometry for two lines to intersect in two places, thus AA' and BB' do not intersect. Again since ∠PKM and ∠QLM are equal, if directed line AA' is parallel to directed line BB', directed line A'A will be parallel to directed line B'B. But the fundamental axiom of hyperbolic geometry states that there exist two distinct lines which are parallel to a given in the two directions, thus AA' and BB' neither intersect nor are parallel. (Sommerville 34)

This theorem implies that if a transversal meets two parallel lines it makes the sum of the interior angles on the side of parallelism less than two right angles.

Theorem 4: *An exterior angle of a triangle is greater than either of the interior opposite angles.*



Proof: Let ABC be a triangle and extend side BC to point D. If the exterior angle $\angle$ACD is not greater than $\angle$ABC, either $\angle ACD < \angle ABC$ or $\angle ACD = \angle ABC$.

Case I: Assume $\angle ACD = \angle ABC$. Then $\angle ABC + \angle ACB = \pi$, and BA, CA would not meet as proved in **theorem 3** above.

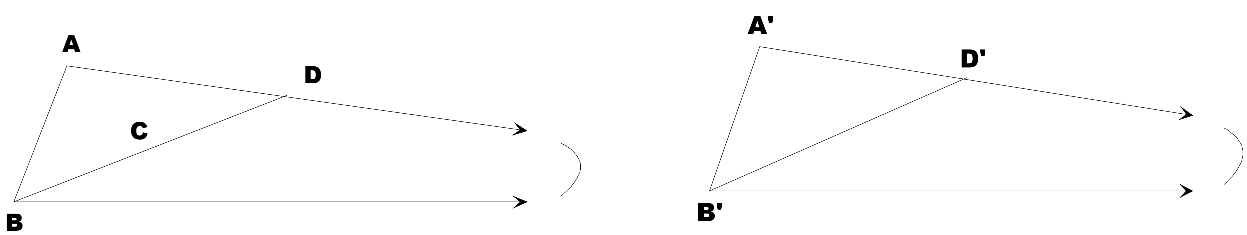
Case II: Assume $\angle ACD < \angle ABC$. Draw a segment joining B to a point A' such that $\angle A'BC = \angle ACD$. BA' lies within the angle ABC and must meet AC by Pasch's axiom, making $\angle A'BC + \angle A'CB = \pi$, which is impossible again by **theorem 3**. (Sommerville 35)

Before going further, it is important to discuss a very useful concept used in hyperbolic geometry which may seem strange at first glance. Two lines that intersect have, by definition, a point in common, whereas parallel lines do not intersect and do not share a point in common. Two parallel lines do have something in common, however, namely that they are parallel to one another. For the sake of convenience two parallel lines are said to intersect at or have in common an ideal point at infinity. All lines parallel in a given sense to a line all meet at an ideal point and constitute a sheaf of lines with an ideal vertex. A line then, besides containing all its actual points contains two additional ideal points, one in each direction, through which pass all lines parallel to it in each direction.

Ideal points are also commonly referred to as points at infinity or infinitely distant points. If this concept seems farfetched, note that the definition of an ordinary point as being at a location without any magnitude is equally puzzling. Both concepts are convenient terms with which to convey relations. To say that two lines meet at infinity is synonymous with saying they are parallel. If we wish to describe a line through a given point O parallel to a given line l in a certain direction, we can refer to the line connecting O to the ideal point of l in that direction. These points at infinity will take on great significance as we proceed, but it is not uncommon in mathematics for concepts originally used for convenience to develop great importance in their own right (Wolfe 71). Ideal points should be treated just like actual points when dealing with any properties or theorems involved with hyperbolic geometry, thus all theorems proved above also hold for triangles and quadrilaterals which may have a vertex at an ideal point. An ideal point is customarily signified by Ω.

The following theorem will make use of triangles with an ideal vertex, or limit triangles.

Theorem 5: *If AB and A'B' are equal, and angle BAΩ is equal to angle B'A'Ω, then angle ABΩ is equal to angle A'B'Ω and the figures are congruent.*



Proof: We proceed by contradiction. Assume ∠ ABΩ and ∠ A'B'Ω are unequal and that ∠ABΩ > ∠A'B'Ω. Construct angle ABC equal to angle A'B'Ω. Thus BC cuts AΩ at a point D. Measure off A'D' on A'Ω' of the same length as AD and draw B'D'. The triangles ABD and

A'B'D' are thus congruent (by Euclid prop.4). Thus angle A'B'D' is equal to angle ABD, which is equal to angle A'B'Ω' by our construction of BD. However ∠A'B'D'= ∠A'B'Ω' implies a contradiction since B'D' intersects A'Ω but B'Ω does not. Thus we conclude that angles ABΩ and A'B'Ω' are equal. (Wolfe 74)

Theorem 6: *The parallel angle Π(h) diminishes as the distance between the lines, h, increases.*

Proof: Suppose AA' and BB' are parallel to MM', and ABM ⊥ MM'. Also assume AM > BM. By theorem 1 above ∠MAA' + ∠ABB' < π, but being on a straight line ∠MBB' + ∠ABB'= π. Thus ∠MAA' < ∠MBB'. (Sommerville 35)

Let us pause a moment here in our formal mathematical proofs to discuss the importance of this relationship between the angle of parallelism and its related distance. Everyone should be familiar with the notion that in Euclidean geometry angles are said to be absolute. This means that angles possess a natural unit of measure (whether that be expressed in radians or degrees) which is intrinsic in the system of geometry. The same holds true for angles in Hyperbolic geometry, as a right angle has a specific definition set forth by the axioms of the geometry. Obviously length does not have that same quality of absoluteness in Euclidean geometry. There are no natural units to measure length intrinsic in the geometry in the mold of right or 180° angles. An arbitrary unit of measurement must be selected which is in no way related to the structure of the geometry. Thus length in Euclidean geometry is said to be relative.

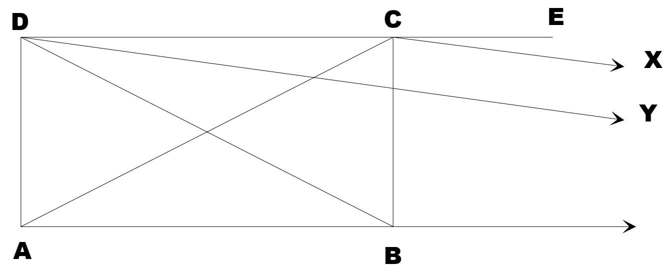
However as **theorem 5** explicitly indicates, in Hyperbolic geometry both angles and lengths are absolute. For every angle of parallelism, $\Pi(h)$, there exists a single corresponding distance h, thus a unit of length can be developed purely by associating a given angle measure to a specific distance (Eves 297). As h approaches 0, the angle of parallelism approaches a right angle, and as h tends to infinity, $\Pi(h)$ approaches 0. Let us now go on to our next set of theorems.

The Saccheri quadrilateral and the Lambert quadrilateral are two very significant geometrical figures named respectively for two of the forerunners in the investigation of non-Euclidean geometry. With a strong grasp of the properties of these figures we will be able to prove some of the most profound theorems of hyperbolic geometry: All triangles have an angle sum less than π, and all similar triangles are necessarily congruent.

Gerolamo Saccheri, an Italian Jesuit priest, perhaps the first man to explore the denial of the Fifth Postulate in hopes of proving it true by *reductio ad absurdum*, employed a particular figure which now carries his name. This figure was an isosceles quadrilateral having two right angles as its base angles . We construct a Saccheri quadrilateral, as seen below, by forming two perpendiculars of equal length at the ends of segment AB and joining the ends of these perpendiculars. The segment AB adjacent to the right angles will be called the base and the opposite side the summit, with its adjacent angles referred to as summit angles. (Wolfe 77)

Theorem 7: *The summit angles of a Saccheri quadrilateral are acute and equal.*

Proof: Triangles DAB and CBA are congruent by Euclid proposition 4. Thus the diagonals AC and BD are equal. It then follows that triangles ADC and BCD are congruent by proposition 8 of Euclid., and thus $\angle ADC = \angle BCD$. Let directed lines CX and DY be the parallels to directed line

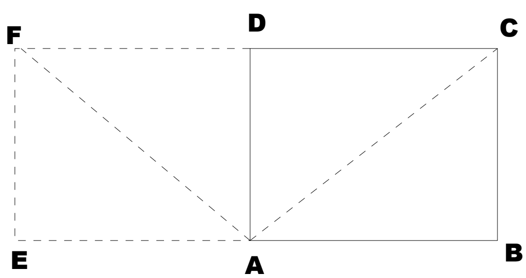


AB through points C and D respectively. By **theorem 4** ∠ECX > ∠EDY. However since their perpendiculars are equal, ∠BCX = ∠ADY. It follows that ∠BCE > ∠ADE, however ∠ADE =∠BCD. Thus it must be the case that ∠BCD is acute. (Eves 299)

J.H. Lambert was another man who nearly missed being hailed the discoverer of non-Euclidean geometry but made significant contributions nonetheless. He studied the quadrilateral having three right angles which we now turn our attention to. Such a trirectangular quadrilateral is called a Lambert quadrilateral.

Theorem 8: *If a quadrilateral has three right angles, the fourth angle is acute.*

Proof: Let ABCD be a Lambert quadrilateral having right angles at A, B, and D.

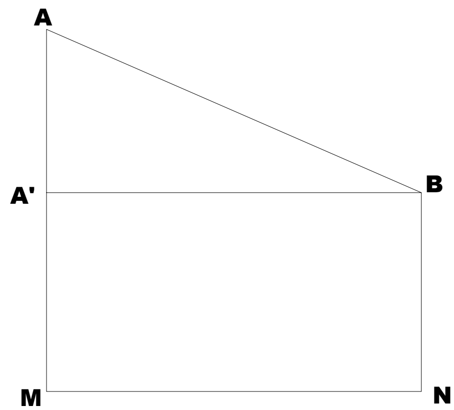


Extend BA through A to point E so that AE = BA. AT E draw EF perpendicular to BE such that EF = BC. Join F to A and D, and draw AC. By the congruence of right triangles FEA and CBA, triangles FAD and CAD are congruent. Thus ∠FDA is a right angle, the points F, D, and C are collinear, and quadrilateral EBCF is a Saccheri quadrilateral. But summit angles of Saccheri quadrilaterals are acute by **theorem 7**. Thus angle C is acute. (Wolfe 78)

Theorem 9: *If AM, BN ⊥ MN and AM > BN, then ∠MAB < ∠NBA.*

Proof: Cut off MA' from MA so that MA' = NB. Then ∠NBA > ∠ NBA' = ∠MA'B > ∠MAB, from **theorem 4**.

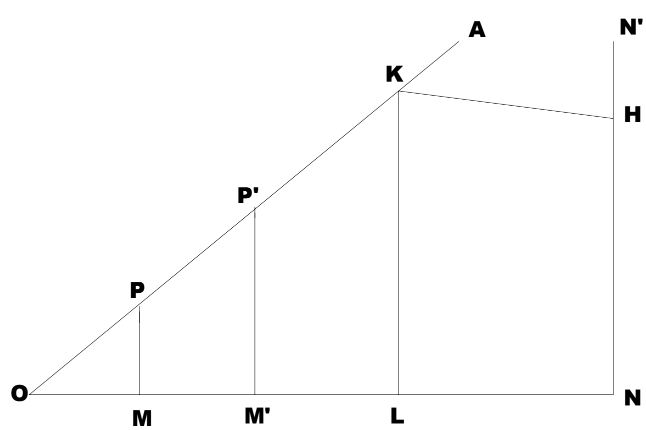
The converse of this would be if ∠MAB < ∠NBA, then AM > BN which will be used later in **theorem 11**.



Theorem 10: *The distance between two intersecting lines increases without limit.*

Proof: Consider two points P, P' on OA such that OP' > OP, and drop perpendiculars PM, P'M' onto ON. The angles M'P'O and MPO are both necessarily acute. Thus ∠M'P'P < ∠MPP', and M'P' > MP by Theorem 3.

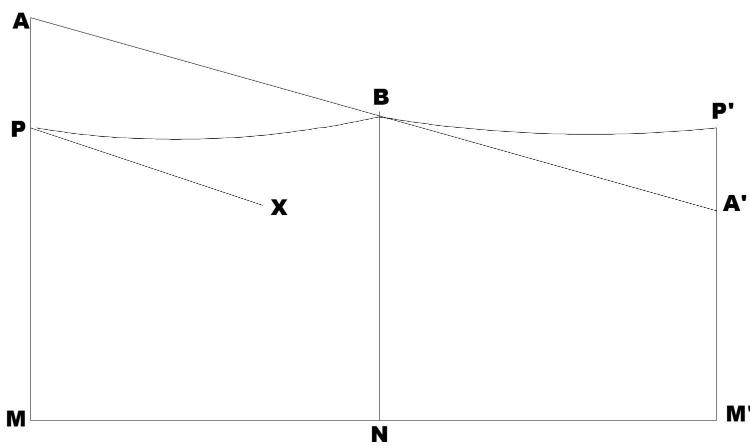
Consider any desired length G. Let ON be the distance corresponding to the parallel-angle ∠NOA and draw NN' ⊥ ON. Thus NN' || OA. Take NH > G, and draw a line HK making the acute angle ∠N'HK. Then HK, lying within ∠OHN', must meet OA in some point K. Draw



KL $\perp$ ON. Since $\angle$KHN is obtuse, $\angle$LKH < $\angle$NHK. Thus LK > NH > G. Hence the perpendicular PM may exceed any given length. (Sommerville 37)

Finally we have enough artillery to prove that fundamental difference in the behavior of parallels in Hyperbolic geometry as compared to Euclidean. This difference is restated in the following two theorems.

Theorem 11: *The distance between two parallels diminishes in the direction of parallelism and tends to zero.*



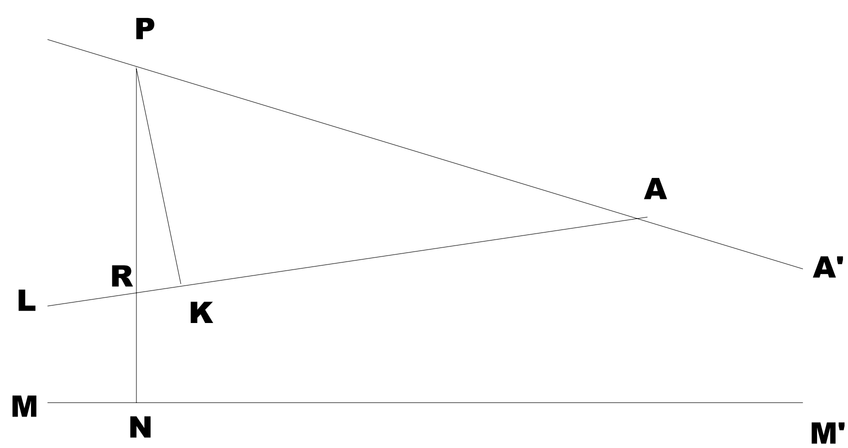
Proof: Let AA' || MM', and let AM, BN be two perpendiculars dropped to MM' from AA' such that B lies on the side of A in the direction of parallelism. The angles $\angle$MAA' and $\angle$NBA' are both acute. Thus $\angle$MAB < $\angle$NBA and NB < MA by the converse of Theorem 6.

Pick any $\varepsilon > 0$ as small as you like and make MP < ε. Draw PB $\perp$ MA. If PX is parallel MM', $\angle$MPX is acute. Thus PB lies within the angle $\angle$APX and in turn meets AA' in some point B, as PX || AA'. Make $\angle$NBP' = $\angle$NBP, BP' = BP, and draw P'M' $\perp$ NM'. By Theorem 1 BP' neither intersects nor is parallel to NM'. BA' must lie within $\angle$M'BP', thus BA' meets M'P' at some point A'. In consequence, M'A' < M'P' < ε. (Sommerville 38)

So we see that parallel lines become closer and closer as they are extended in the direction of parallelism. Since we can always find an M'A' less than any ε. Now let's consider what happens on the left side of our figure as AA' extends in the direction opposite to that of parallelism.

Theorem 12: *The distance between two parallels increases in the direction opposite to that of parallelism.*

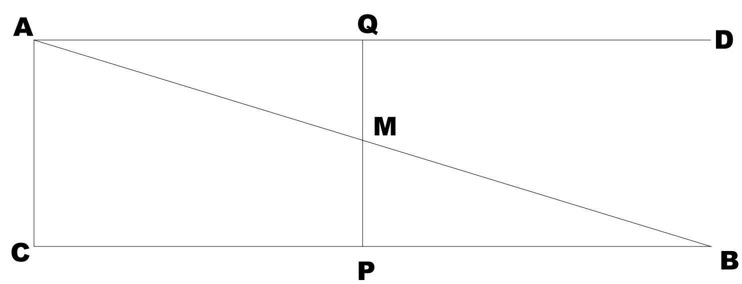
Proof: Let AL || M'M. Consider any point P on A'A, and drop a perpendicular PN onto M'M that cuts AL at R, then draw PK ⊥ AL. Thus PN > PR > PK and PK, the distance from AL to P, can be as long as we like. It just depends on our choice of P, and hence PN can exceed any length (Sommerville 39).



Thus we have arrived at the fundamental difference between parallels in Hyperbolic versus Euclidean geometry. Parallel lines are not equidistant lines in Hyperbolic geometry, they are instead said to be asymptotic in their behavior as they become arbitrarily close in the direction of parallelism and arbitrarily divergent in the opposite direction. It is common to then to speak of parallel lines as meeting at infinity, where their intersection is said to be of angle zero. Behavior of parallels in hyperbolic geometry is thus described as asymptotic (Sommerville 41).

We now have all the necessary theorems to prove some of the more interesting results in Hyperbolic geometry. The first being that all triangles have angle sum less than π. We start by proving this fact for all right triangles.

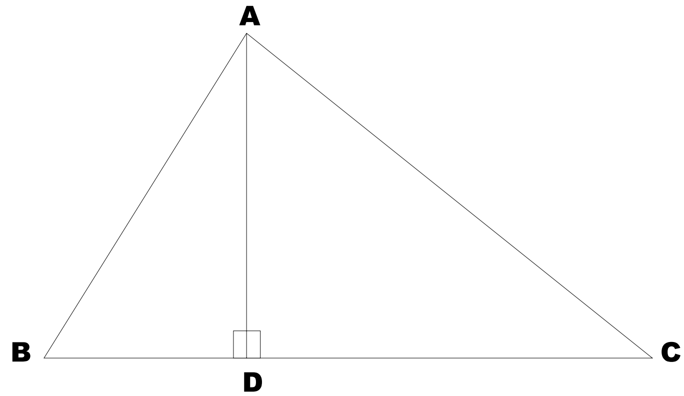
Theorem 13: *All right triangles have angle sum less than π.*



Proof: Let ABC be any right triangle with the right angle at C. We know then that the other two angles must be acute. Construct ∠BAD as equal to ∠ABC. At the midpoint, M, of AB draw segment MP perpendicular to CB. Draw MQ cutting AD such that AQ = PB. The triangles MBP and MAQ will be congruent, and consequently ∠AQM is a right angle, points Q, M, and P are collinear, and ACPQ is a Lambert quadrilateral with an acute at A. Thus the sum of the acute angles of triangle ABC is less than one right angle and the sum of all three angles is less than two right angles.

Theorem 14: *The sum of the angles of every triangle is less than two right angles.*

Proof: Since we've already proved this for a right triangle, we assume ABC has no right angles. We know that every triangle has at least two acute angles, so let's call the angles at B and C acute. Let's drop a line of altitude from A to D on BC. By definition of an altitude, ABC has been divided into the two right triangles ADB and ADC.



Since the sum of ∠ABD and ∠BAD is less than one right angle, likewise for ∠ACD and ∠CAD, the sum of the angles of triangle ABC is less than two right angles, or π.

Since any quadrilateral can be divided into two triangles, the sum of the angles of a quadrilateral is always less than four right angles, or 2π. Thus rectangles as we know them do not exist in Hyperbolic geometry. This brings us to an important notion in the study of Hyperbolic geometry known as the defect. The difference between π or 180° and the angle sum of a triangle is known as the triangle's defect. Similarly the difference between the angle sum of a quadrilateral and 2π, 360°, is known as its defect. The defect is intimately related to the concept of area in Hyperbolic geometry, and specifically this relationship accounts for the existence of an upper bound for the area of a triangle.

It turns out that when we construct triangles in Hyperbolic geometry the smaller the triangle, the smaller its defect. That is, as the perimeter of a triangle becomes less and less, the angle sum of the triangle becomes closer and closer to 180°. Consequently when dealing with small lengths, Hyperbolic triangles closely approximate Euclidean triangles. This supports the notion that in localized spaces with uniform gravitation Hyperbolic and Euclidean geometry coincide without great difficulty. Thus we continue to use Euclidean geometry and teach it to our children as it is extremely useful and accurate for everyday practical needs which deal with relatively small distances. Variance in the predictions and assumptions made by Hyperbolic and

Euclidean geometry only comes into play when we begin to describe huge expanses of space and time in areas of astrophysics such as General Relativity Theory.

Likewise the defect of a triangle increases as the length of its sides increase. Thus as the sides of a triangle approach infinity, the angles of that triangle approach zero and the defect becomes very close to 180°. Its also important to note that a triangle's defect is an additive property such as its area, meaning that if we divide a given triangle into several smaller ones the original triangle's defect will equal the sum of its constituent triangles' defects. It follows that since a quadrilateral can always be divided into two triangles, we can ascribe a defect to a quadrilateral in Hyperbolic geometry as the sum of the defects of its two inner triangles. More generally, any polygon can be divided into a finite number of non-overlapping triangles, so the defect of a polygon is defined as the sum of the defects of the triangles it can be divided into. This defect remains constant regardless of the different ways that the polygon may be divided.

Since the defect of a triangle increases in as the perimeter increases, intuitively it would make sense for the defect to be directly proportional to the area of triangle or other polygon. This turns out to be precisely the case, as the Euclidean formula, Area = ½ bh, must be abandoned as A Hyperbolic triangle may have three different values for ½ bh for its three respective sides. Thus the formula for the area of a Hyperbolic triangle must involve the defect, as the defect increases and decreases as the space being described becomes greater or less.

In general an area function in any geometry should have the following properties:

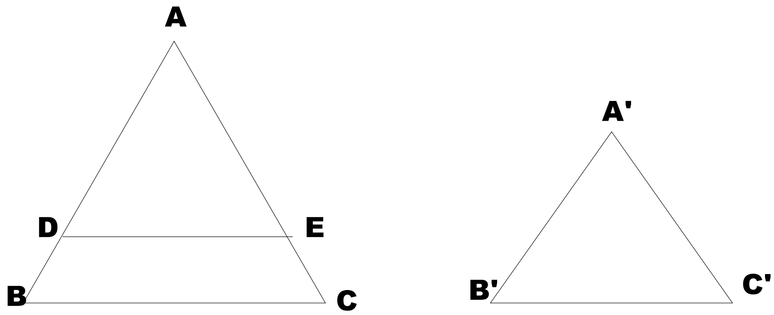
1.) It must be able to determine the area of any polygonal region.
2.) It must give a value greater than zero for any polygonal region,
3.) It must hold that if two triangular regions are congruent, their areas are equal.
4.) It must ensure that the area of the union of two non-overlapping polygonal regions is the sum of their areas.

It can proven that every function that satisfies the condition of having the above properties has the form of a simple constant times the defect. We shall denote this constant as k and write the defect as d, yielding A=kd. (Moise 270). This constant k depends upon the unit of measure, that is it depends upon what triangle is said to have area equal to 1 (Greenberg 265).

The constant k is what transforms the units of the side lengths to units of area. A remarkable consequence of this formula for area may have already occurred to the reader. If area is the product of a constant and the defect, then just as the defect has an upper bound, 180°, so must the area, k(180). Admittedly this is a very difficult concept to swallow at first since in a Euclidean frame of mind it seems common sense that the area of a triangle can become greater and greater without limit. It is amazing what far-reaching consequences a simple alteration of one postulate can make, but nonetheless it is a necessary result of the Hyperbolic system that a triangle's area is proportional to its angle-sum, which has an upper bound. Once in a letter to Bolyai, Gauss noted that if there exists a triangle of maximum area it must be the limiting form of a triangle. This triangle would have all three vertices at ideal points, with the angle at those points being zero, and must then consist of a given line and the two parallel lines to that line in both of the opposite senses (Wolfe 128).

In concluding the examination of the basic properties of the two-dimensional Hyperbolic plane, we encounter another instance wherein the relationship between angle-sum, perimeter, and area of Hyperbolic triangles leads to an interesting result. The fact is that in Hyperbolic geometry similar triangles do not exist. That is, the only similar triangles, triangles having the same angle to side proportions, that exist are the trivial cases where the two triangles are congruent.

Theorem 15: *If the three angles of one triangle are equal, respectively, to the three angles of a second, then the two triangles are congruent.*



Proof: Let angles A, B, C of triangle ABC be equal, respectively, to angles A', B', C' of triangle A'B'C'. If any pair of corresponding sides are equal then obviously the triangles will be congruent. Thus we assume that two corresponding sides, AB and A'B' are not equal. Assume for the sake of our figure that AB is greater than A'B'. Cut off on AB the segment AD so that AD = A'B' and cut off AE on segment AC so that AE = A'C' as shown above. AE must be less than AC which we can see by examining the alternatives. If AE = AC then $\angle$BCA and $\angle$DCA would be equal. But this cannot be as we are supposing AD to be less than AB. Likewise if AE were greater than AC, an exterior angle of the triangle would be equal to one of the opposite interior angles which has already been shown to be impossible. Since triangles ADE and A'B'C' are congruent (SAS), it is evident that quadrilateral BCED has an angle sum equal to four right angles. However we have already proved that this is impossible, thus it must be the case that AB = A'B' and that the triangles are congruent.

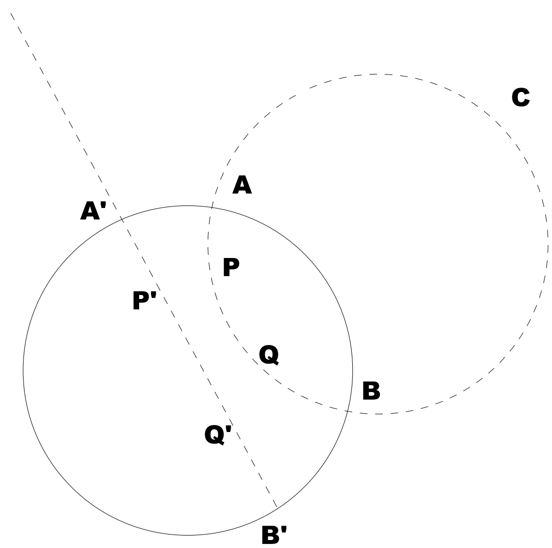
Along the same lines as the previous discussion of ideal points, there exists another concept of convenience to describe non-intersecting lines, also sometimes referred to as hyperparallels. As we have noted, in Hyperbolic geometry a line intersects a given line if it lies within the angle of parallelism to a parallel of that given line, and does not intersect (is hyperparallel to) that given line if it lies outside of the angle of parallelism. Arguing in much the same vein as we did for ideal points, though non-intersecting lines do not share an actual point in common they do have something in common. That is namely, they share a common perpendicular. Thus by convention non-intersecting lines are said to have in common, or even intersect at, an ultra-ideal point. So all lines perpendicular to a given line are regarded as having an ultra-ideal point in common and form sheaf of lines having this ultra-ideal point for a vertex. Thus given two hyperparallel or non-intersecting lines they determine an ultra-ideal point the sheaf of lines having that ultra-ideal point as its vertex contains all lines which cut the perpendicular to the two given lines at right angles. Thus an ultra-ideal point always describes a partnering representative line, so that every line through the point is perpendicular to that line and vice versa (Wolfe 86).

It is often very difficult to get one's mind around, so to speak, the validity of arguments in Hyperbolic geometry as we are so conditioned to think in Euclidean terms. Because Euclidean geometry is so familiar to us, it is common and very useful to invoke models of Hyperbolic geometry using Euclidean representations. One of the purposes of such an endeavor is that since all elements of Hyperbolic geometry are represented in Euclidean terms, if Euclidean plane geometry is consistent it confirms that Hyperbolic plane geometry is as well. Thus these models can be considered as interpretations of Hyperbolic geometry in Euclidean terms. There are five or six very common such models and there are many more besides. We will examine one of the most common ones, the interior of a disk model, which is particularly nice because of its simplicity. This model was developed by Henri Poincare, a brilliant mathematician and philosopher, and is thus often referred to as the Poincare model for Hyperbolic plane geometry. It is important to remember that the lines, angles and distances in these models are not what these entities "really are" in Hyperbolic geometry. These concepts can be defined any way we like, and if the relations among these new definitions accord with the relationships set forth in Hyperbolic geometry, we have a working model.

Consider a fixed circle Σ in the Euclidean plane. For convenience's sake we assume this circle is centered at the origin with radius 1. Let C be any circle orthogonal to circle Σ, orthogonal here means that their tangent lines are perpendicular. The following definitions translate the basic players of Hyperbolic into their representatives in the disk model.

Point: Any Euclidean point on the interior of circle Σ. We let Ω describe the set of all points.

Line: A line can be either (1) the intersection of Ω and an orthogonal circle C or (2) the intersection of Ω and a diameter of Σ.



Note that lines and points thus defined satisfy the axioms of Hyperbolic geometry, such as two points lie on exactly one line. One axiom which may not seem so obviously satisfied by this model is that a line segment with two endpoints may be extended further in either direction. How can a line be extended in either direction if they are bound by the perimeter of Σ? The answer lies in the fact that the lines defined in this model form open intervals, thus no matter how close an endpoint may be to the edge of Σ, it can always be a little closer without reaching it.

The distance function for the Poincare model must exhibit the properties of any common distance function. Being namely the properties of being positive definite, symmetric, and upholding the triangle inequality. In addition, Euclid's third axiom requires that it be possible to construct a circle of any radius around any given point, so the distance function must be continuous and onto the range of all real numbers. (Moise 427)

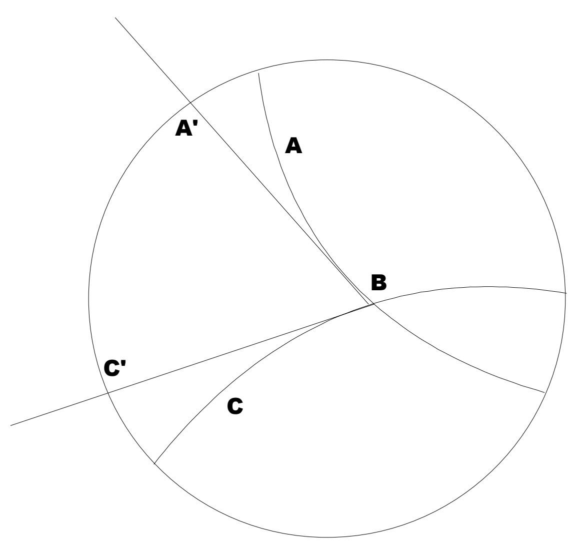
Distance: Let P and Q be two points within Σ. These points determine a unique line that approaches the boundary of Σ at the two points A and B as shown in the figure above. (Note A and B are not actual points as they lie on the boundary circle.) Let |PA|, |PB|, |QA|, and |QB| be the values of the Euclidean distance metric between those respective points. The distance between point P and Q is defined as:

$$d(PQ) = \ln \frac{PA \ / \ PB}{QA \ / \ QB}$$

The proof of this is omitted because of its length but the reader should note that the distance formula has such a peculiar nature because it must provide for some peculiar characteristics of the Poincare model. Points near the boundary circle which may appear quite close can in fact be great distances apart.

Circle: The set of all points equidistant from a given point.

Angle Measure: Given three points A, B, C, construct the Euclidean rays BA', BC' tangent to lines BA and BC respectively at point B. The angle measure of ∠ABC is defined as equal to the measure of Euclidean angle ∠A'BC'.



A natural question that must arise as a result of studying and contemplating these differing systems of geometry is: Which geometry describes the way physical space *really* is? Though an obvious question, an answer does not present itself so readily. The only difference in the axiomatic formulation of Euclidean versus the Hyperbolic geometry is the postulate stating there are two lines parallel to a given line through a given point. Adopting this postulate maintains Hyperbolic geometry's consistency with all other postulates of Euclidean geometry. So in determining which geometry better describes actual physical space, the only ground on which to judge is to see which more closely agrees with facts observed in physical space. It seems there needs to be a crucial empirical test of which conception describes reality best. An example of such a test would be to measure the angles of enormous triangle in space and determine whether they add up to 180°. So far no triangle has been found in the physical world capable of being measured, which is large enough so that the defect from 180° can be accounted for without the possibility that it comes simply from measurement error (measurement error is always present to some degree, and goes up as the distances increase) (Eves 315).

Indeed it may be impossible to ever confirm such a crucial test as we are limited by spaces intimate link with matter and limitations it puts on us as observers. It is perhaps better then, to speak of which geometry is more useful or convenient, rather than more true. Obviously Euclidean geometry is the most useful system for common architectural drafting, bridge building, and other practical engineering purposes on Earth. However when Einstein realized that neither Hyperbolic nor Euclidean geometry was adequate for his purposes in his general theory of relativity, he employed a Riemannian geometry which properties depending on the concentration of mass. From a philosophical standpoint one could hold that this then would be the closest geometry of how the universe "really is", but could not prove it. Likewise Hyperbolic geometry has been found to be best for describing the space humans perceive visually. Again a philosophical debate can arise as to whether space as we visually experience it is what it means to say space "as it really is."

Patrick Heelan in *Space-Perception and the Philosophy of Science* argues for the notion that humans visually experience the world in hyperbolic geometry, not in Euclidean terms as has been previously assumed. His evidence for this thesis comes from examples of everyday phenomena that everyone can relate to and from some well-known visual illusions which seem to be the effect of hyperbolic vision. Everyday phenomena which suggest hyperbolic vision are mainly due to the existence of *near zones* and *distant zones*. At a certain distance, d, away from the visual observer, an object's apparent visual size and shape coincide with its actual size and shape. The near zone extends to a distance of about 2d away from the observer and within it our visual perception is very consistent with physical reality. However parallel lines will appear to diverge if they are closer than d to us. The reader may test this phenomenon by holding a small rectangular card horizontally in front of their eyes. Focus on the center of the far edge of the card and then move it slowly away from the eyes. The two sides of the card will appear to diverge at first but the observer should be able to locate a distance at which there is a turning point and the lines will begin to appear to converge.

The distant zone is the part of our visual field which extends from the distance $2d$ to the limit of possible vision, the horizon sphere. The horizon sphere is the theoretical limit of visibility, and can be thought of as the spherical surface with the viewer at its center, on which Euclidean infinity is mapped. The general characteristic of the distant zone is an apparent shallowness of depth. Objects in the distant zone display a noticeable telephoto effect. They appear closer to the observer than they physically are, depth between distant objects is foreshortened, and as a result of this loss of depth all surfaces appear to face the observer frontally though in reality they may be angled to some other direction. Horizontal lines in a plane thus appear to us as diverging up to a certain distance, and then beginning to converge until they finally meet at the horizon sphere (Heelan 58).

By the preceding discussion it should be apparent that our visual experience of physical lines which are Euclidean parallels transforms them into hyperbolic parallels. This has been an elementary sketch at best of the arguments for regarding visual space as hyperbolic, and the interested reader should refer directly to the source for an in-depth explanation. However a brief description is relevant to this paper as the reader can see some concrete applications of hyperbolic geometry and its significance to other fields of science.

APPENDIX: The first twenty-eight propositions of Euclid, Book 1

1. On a given finite straight line to construct an equilateral triangle.
2. To place at a given point (as an extremity) a straight line equal to a given straight line.
3. Given two unequal straight lines, to cut off from the greater a straight line equal to the less.
4. If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.
5. In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.
6. If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.
7. Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former respectively, namely each to that which has the same extremity with it.
8. If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.
9. To bisect a given rectilinear angle.
10. To bisect a given finite straight line.
11. To draw a straight line at right angles to a given straight line from a given point on it.
12. To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.
13. If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.
14. If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.
15. If two straight lines cut one another, they make the vertical angles equal to one another.
16. In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.
17. In any triangle two angles taken together in any manner are less than two right angles.
18. In any triangle the greater side subtends the greater angle.
19. In any triangle the greater angle is subtended by the greater side.
20. In any triangle two sides taken together in any manner are greater than the remaining one.
21. If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.
22. Out of three straight lines, which are equal to three given straight lines, to construct a triangle; thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one.
23. On a given straight line and at a point on it to construct a rectilinear angle equal to a given rectilinear angle.
24. If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.

25. If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other.
26. If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle equal to the remaining angle.
27. If a straight line falling on two straight lines make the alternative angles equal to one another, the straight lines will be parallel to one another.
28. If a straight line falling on two straight lines make the exterior angle equal to the interior angle and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.

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Senior Exercise

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