Metrizable Topologies

Hannah Ross
Senior Exercise in Mathematics
Kenyon College

Fall 2009

1 Introduction

"So, young lady, the way forward is sometimes the way back."
–The Wiseman, Jim Henson's The Labyrinth

"It's going to take a while to get a metric for this."
–Richard Doherty

Mathematics teachers are the world's biggest plot-spoilers. Talk about giving away the endings – as students, we learn to add and subtract over a decade before we get the definition of a binary operation; we talk about limits long before we're introduced to limit points; we measure lengths from the time we hit first grade without ever knowing what it means for a function to be a metric, or that there are infinitely other ways we could define the distance from Point A to Point B. And this is *okay* – in fact, it's ideal. Mathematics is all about turning to the last chapter first, it's about knowing where you're going so that you can figure out how to get there. Then, once we understand how the story ends – what sort of statements hold true in a given environment – the next question is, how did we get here? What would it look like if all of these properties we take as givens did *not* hold?

Spending the majority of our early academic lives in subsets and subspaces of $\mathbb{R}^n$, and growing up in a world that seems, for all intents and purposes, comparable to $\mathbb{R}^3$, it becomes second nature to assume that, at the very least, we should be able to measure the distance from Point A to Point B. However, there exist an infinitude of spaces that are *not* metrizable. In this paper, we will begin by looking at general topological spaces and work towards an understanding of what it takes to guarantee that a given topological space will be metrizable. In particular, we will be building up to a proof of Urysohn's Metrization Theorem, which states that every second countable, regular, T_1 space is metrizable.

(A note on methodology: for the most part, I used Michael Starbird's notes to look up definitions, and for a general outline of the proofs and concepts leading up to Urysohn's Metrization Theorem. Otherwise, I tried to write most of my proofs on my own. Only the last two proofs rely heavily on hints from published versions of the proofs, and these will be cited accordingly.)

2 What Is a Topology?

One of the reasons the study of mathematics tends to work backwards is that, paradoxically, we need the more advanced concepts to describe the earlier ones. This is why, in his article "Elementary Point Set Topology," R. H. Bing suggests, "Perhaps it is best not to define topology to people who do not already know what it is" (28). Nevertheless, as this would lead to a rather dead end educational methodology, we will take a run at defining topology anyway, at the risk of temporary confusion.

Definition 2.1 Suppose X is a set. Then $\mathcal{T}$ is a **topology** for X if and only if $\mathcal{T}$ is a collection of subsets of X such that:

1. $\emptyset \in \mathcal{T}$,
2. $X \in \mathcal{T}$,
3. if $A \in \mathcal{T}$ and $B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,
4. if $\{A_\alpha\}_{\alpha\in\lambda}$ is any collection of sets each of which is in $\mathcal{T}$, then $\cup_{\alpha\in\lambda} A_\alpha \in \mathcal{T}$.

When you combine a set and a topology for that set, you get a topological space.

Definition 2.2 A **topological space** is an ordered pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology for X.

The best way to understand topological spaces is to take a look at a few examples. The most basic topology for a set X is the **indiscrete** or **trivial topology**, $\mathcal{T} = \{\emptyset, X\}$. We see that this fulfills all of the requirements of Def. 2.1 – it contains the empty set and X, as well as the intersection and union of those two elements. We can think of this as a minimalist topology – it meets the requirements with nothing extra.

At the other end of the spectrum, we have the **discrete topology**, $\mathcal{T} = \mathcal{P}(X)$, the power set of X. This means that any possible combination of elements in X is an element of $\mathcal{T}$. This will certainly give us X and the empty set in our topology, as well as any possible intersection or union of sets in X.

Another common topology is the **finite complement topology** for a set X, in which a subset U of X is open if and only if $U = \emptyset$ or $U^{\mathcal{C}}$ is finite.

A topology we will be particularly interested in is the **usual topology on** $\mathbb{R}^n$, that is, a subset U of $\mathbb{R}^n$ belongs to $\mathcal{T}$ if and only if for each point p of U there is a positive number ϵ such that $\{x : d(p, x) < \epsilon\}$ is a subset of U. Here, d refers to the **usual metric** on $\mathbb{R}^n$:

$$d(x, y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}.$$

Perhaps the first step to understanding how topologies behave is and examination of open sets.

Definition 2.3 If $(X, \mathcal{T})$ is a topological space, then U is an **open set** in $(X, \mathcal{T})$ if and only if $U \in \mathcal{T}$.

This means that openness depends on the topology on a set, rather than just the set itself. So, we may have a subset of X which is open under one topology and not under another. Consider the set of integers. Let $\mathcal{T}_1 = \{\emptyset, \{x \in \mathbb{Z} : x \text{ is odd}\}, \{x \in \mathbb{Z} : x \text{ is even}\}, \mathbb{Z}\}$ and $\mathcal{T}_2 = \{\emptyset, \{x \in \mathbb{Z} : x \text{ is } 0 \bmod 3\}, \{x \in \mathbb{Z} : x \text{ is } 1 \bmod 3\}, \{x \in \mathbb{Z} : x \text{ is } 2 \bmod 3\}, \{x \in \mathbb{Z} : x \text{ is } 0 \text{ or } 1 \bmod 3\}, \{x \in \mathbb{Z} : x \text{ is } 0 \text{ or } 2 \bmod 3\}, \{x \in \mathbb{Z} : x \text{ is } 1 \text{ or } 2 \bmod 3\}, \mathbb{Z}\}$. We could show that each of these satisfy the characteristics of a topology. In the space $(X, \mathcal{T}_1)$, the set of even integers is in the topology, and therefore open, but in $(X, \mathcal{T}_2)$, this set does not appear in the topology and is *not* open.

Now that we have defined open sets, let's consider what it takes for a set to be closed.

Definition 2.4 Let $(X, \mathcal{T})$ be a topological space with $A \subseteq X$ and $p \in X$. Then:

1. p is a **limit point** of A if and only if for each open set U containing p, $(U \setminus \{p\}) \cap A \neq \emptyset$. Notice that p is not necessarily an element of A.

2. If $p \in A$, but p is not a limit point of A, then p is an **isolated point** of A.

3. The **closure** of A (denoted $\bar{A}$) is A together with all its limit points.

4. The set A is **closed** if and only if A contains all its limit points – if $\bar{A} = A$.

As in a metric space, a closed set as a set containing all of its limit points. However, limit points are defined slightly differently in a general topological space than in a metric space. We will take a closer look at this momentarily. However, first, we will need to establish a few basic rules about the behavior of open and closed sets in general topological spaces.

Theorem 2.5 For any topological space $(X, \mathcal{T})$ and subset A of X, $\bar{A}$ is closed.

Proof:

Let p be a limit point of $\bar{A}$. If $p \in A$, then $p \in \bar{A}$ and we are done.

So, suppose instead that $p \notin A$. Let U be an open set containing p. We want to show that U must also contain a point of A in order to conclude that p must be a limit point for A and thus an element of $\bar{A}$.

By Def. 2.4, since U is an open set containing p, $(U \setminus \{p\}) \cap \bar{A} \neq \emptyset$. So there exists $q \in (U \setminus \{p\}) \cap \bar{A}$. So $q \in \bar{A}$. If $q \in A$, we have what we want. If $q \notin A$, q is a limit point of A. Therefore, since U is an open set containing q, $(U \setminus \{q\}) \cap A) \neq \emptyset$. So there exists some point $r \in U$ such that $r \in A$. Since we have assumed $p \notin A$, $r \neq a$. So we have $U \setminus \{p\} \cap A \neq \emptyset$.

So, we may conclude that all open sets containing p will also contain an element of A. Therefore, by Def. 2.4, p is a limit point for A. So $p \in \bar{A}$. Since p was an arbitrary limit point for $\bar{A}$, we see that $\bar{A}$ contains all of its limit points. So $\bar{A}$ is closed. ∎

This will be useful later on. We will be taking closures of sets, and it will be important to be sure that these closures are indeed closed sets. We will also be taking complements of sets, and the next theorem deals with these.

Theorem 2.6 Let $(X, \mathcal{T})$ be a topological space. Then a subset A of X is closed if and only if $A^{\mathcal{C}}$ is open.

Proof:

$(\Rightarrow)$ Suppose A is closed. Let $p \in A^{\mathcal{C}}$. Since $p \notin A$ and A is closed, p cannot be a limit point for A. So there exists some open set U_p such that $p \in U$ and $U \setminus \{p\} \cap A = \emptyset$. Since p was not in A to begin with, $U \cap A = \emptyset$. We can construct such a set U_q for all $q \in A^{\mathcal{C}}$. Now, let $V = \cup_{q \in A^{\mathcal{C}}} U_q$. Each $q \in A^{\mathcal{C}}$ will be in its own U_q and thus an element of the union, so $A^{\mathcal{C}} \subseteq V$. Furthermore, $U_q \cap A = \emptyset$ for all $q \in A^{\mathcal{C}}$, so if $r \in V$, $r \notin A$, so $r \in A^{\mathcal{C}}$. Thus $V \subseteq A^{\mathcal{C}}$. Therefore, we may conclude that $A^{\mathcal{C}} = V$. So $A^{\mathcal{C}}$ is a union of open sets. Thus, by Def. 2.1, $A^{\mathcal{C}}$ is open.

$(\Leftarrow)$ Suppose $A^{\mathcal{C}}$ is open. Let p be a limit point of A. By definition of complement, $(A^{\mathcal{C}}) \cap A = \emptyset$. So $(A^{\mathcal{C}}) \setminus \{p\} \cap A = \emptyset$. Therefore, by the definition of a limit point, $p \notin (A^{\mathcal{C}})$. But we know p is in X. So $p \in A$. Thus, A contains all of its limit points – it is closed. ∎

So the complement of an open set is closed and vice versa. Note that this does not mean that open and closed are opposites. A set – most notably the set X itself – can be both open and closed. Its complement – in this case the empty set – will therefore also be both open and closed. Additionally, there are sets that are neither open nor closed with complements of this same ilk. Here is another useful fact about closures:

Theorem 2.7 Let $(X, \mathcal{T})$ be a topological space. For subsets A, B of X, if $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.

Proof:
Let A and B be subsets of X such that $A \subseteq B$. Let $x \in \bar{A}$. If $x \in A$, we get $x \in A \subseteq B \subseteq \bar{B}$ and we're finished. So, suppose $x \notin A$. Then x is a limit point for A. We want to show that x is also a limit point for B. Let U be an open set containing x. Then, since x is a limit point for A, there exists some element y in $U \setminus \{x\} \cap A$. Since $A \subseteq B$, $y \in U \setminus \{x\} \cap B$. Since U was arbitrary, all open sets U containing x, $U \setminus \{x\} \cap B \neq \emptyset$. So x is a limit point for B, and thus $x \in \bar{B}$. We conclude that $\bar{A} \subseteq \bar{B}$. ∎

Finally, we want to use our knowledge about intersections and unions of open sets to derive similar properties for the intersections and unions of closed sets.

Theorem 2.8 The union of finitely many closed sets in a topological space is closed.

Proof:
Let $\{U_i\}_{i \leq n}$ for some $n \in \mathbb{N}$ be a finite collection of closed sets in a topological space $(X, \mathcal{T})$. By Thm 2.6, for all $i \leq n$, $U_i^{\mathcal{C}}$ is open. Therefore, as the intersection of finitely many open sets in $\mathcal{T}$, $\cap_{i \leq n} U_i^{\mathcal{C}}$ is open. By DeMorgan's Laws, $\cap_{i \leq n} U_i^{\mathcal{C}} = (\cup_{i \leq n} U_i)^{\mathcal{C}}$. So, since its complement is open, by 2.6 $\cup_{i \leq n} U_i$ is closed. ∎

Note that we will not necessarily get closed sets from unions of infinitely many closed sets. We can find an example of this in $\mathbb{R}$ under the usual topology. Consider the union of all singleton sets containing points in the interval $(0, 1)$:

$$\cup_{x \in (0,1)} \{x\} = (0, 1).$$

These are all closed sets when taken individually, yet the union gives us the whole open interval, which is *not* a closed set.

Theorem 2.9 Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed subsets of a topological space. Then $\cap_{\alpha \in \lambda} A_\alpha$ is closed.

Proof:
Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed subsets of a topological space. Then, by Thm 2.6 $\{A_\alpha^{\mathcal{C}}\}_{\alpha \in \lambda}$ is a collection of open subsets of X. So all of the A_α's are in $\mathcal{T}$. So, by Part 4 of Def 2.1, $\cup_{\alpha \in \lambda} A_\alpha^{\mathcal{C}}$ is also in $\mathcal{T}$ and thus open. By

DeMorgan's Laws, $\cup_{\alpha\in\lambda}A_\alpha^{\mathcal{C}} = (\cap_{\alpha\in\lambda}A_\alpha)^{\mathcal{C}}$. So, since its complement is open, by 2.6 $\cap_{\alpha\in\lambda}A_\alpha$ is closed. ∎

These theorems show us that, on a basic level, general topological spaces function in the ways we would expect from our experience with open and closed sets in metric spaces. They serve both as a nice warm-up for working with non-metrically defined open sets and as essential building blocks for the theorems ahead.

3 Bases and Metrization

If we have any open set in a metric space, we know we can build it out of open balls. In a general topological space, this concept of a set of basic building blocks for a topology is called a basis.

Definition 3.1 Let $\mathcal{T}$ be a topology on a set X and let $\mathcal{B}$ be a subset of $\mathcal{T}$. Then $\mathcal{B}$ is a **basis** for the topology $\mathcal{T}$ if and only if every element of $\mathcal{T}$ is the union of elements in $\mathcal{B}$.

We can think of a basis $\mathcal{B}$ as a "compressed" version of a topology, $\mathcal{T}$ – it's a subset of $\mathcal{T}$ that contains all of the necessary pieces to construct $\mathcal{T}$. In order to "expand" $\mathcal{B}$ into $\mathcal{T}$, we take all possible unions of sets in $\mathcal{B}$. This means that the topology generated by a given basis of a set X is unique. We will demonstrate this in a brief proof.

Theorem 3.2 Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X, and let $\mathcal{B}$ be a basis for both $\mathcal{T}_1$ and $\mathcal{T}_2$. Then $\mathcal{T}_1 = \mathcal{T}_2$.

Proof:

Let $A \in \mathcal{T}_1$. By the definition of a basis, we know that $A = \cup_{\alpha\in\lambda}B_\alpha$ where the B_α's are all sets in $\mathcal{B}$. Since $\mathcal{B}$ is a basis for $\mathcal{T}_2$, $\mathcal{B} \subseteq \mathcal{T}_2$. So we must have $B_\alpha \in \mathcal{T}_2$ for all $\alpha \in \lambda$. As a union of open sets in $\mathcal{T}_2$, $\cup_{\alpha\in\lambda}B_\alpha = A$ will also be in $\mathcal{T}_2$ by the definition of a topology. So we have $\mathcal{T}_1 \subseteq \mathcal{T}_2$. To show that $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we can simply switch $\mathcal{T}_1$ and $\mathcal{T}_2$ in the above argument. So we get $\mathcal{T}_1 = \mathcal{T}_2$. ∎

Thus, the topology arising from a basis $\mathcal{B}$ is unique. However, a given topology $\mathcal{T}$ for a set X can have many bases. To give an example of this, we will first need the following characterization of bases to get a sense for how we can choose a basis out of a set X.

Theorem 3.3 Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for a topology for X if and only if the following conditions hold:

1. $\emptyset \in \mathcal{B}$,
2. for each point $p \in X$, and every open set W containing p there is a set $U \in \mathcal{B}$ with $p \in U \subseteq W$, and
3. if U and V are sets in $\mathcal{B}$ and $p \in U \cap V$, there is a set W in $\mathcal{B}$ such that $p \in W \subseteq (U \cap V)$.

Proof:
($\Rightarrow$) Suppose $\mathcal{B}$ is a basis for a topology for X.

1. $\underline{\emptyset \in \mathcal{B}}$. Since the empty set must be in the topology, we must have $\emptyset = \cup_{\alpha\in\lambda} B_\alpha$ where $B_\alpha \in \mathcal{B}$ for all $\alpha \in \lambda$. The only way for this to happen is for the empty set to be an element of $\mathcal{B}$.
2. $\underline{\text{For each point } p \in X \text{ and open set } W \text{ containing } p\text{, there is a set } U \in \mathcal{B} \text{ with } p \in U \subseteq W}$. Let $p \in X$ and let W be an open set containing p. Since W is open, $W \in \mathcal{T}$. So by the definition of basis, $W = \cup_{\alpha\in\lambda} B_\alpha$ where $B_\alpha \in \mathcal{B}$ for all $\alpha \in \lambda$. Since $p \in W$, $p \in \cup_{\alpha\in\lambda} B_\alpha$, meaning $p \in B_\alpha$ for some $B_\alpha \in U$. So we get $p \in B_\alpha \subseteq W$.
3. $\underline{\text{If } U \text{ and } V \text{ are sets in } \mathcal{B} \text{ and } p \in U \cap V\text{, there is a set } W \text{ in } \mathcal{B} \text{ such that } p \in W \subseteq (U \cap V)}$. Suppose U and V are sets in $\mathcal{B}$. Then U and V must be in any topology that has $\mathcal{B}$ for a basis (since a basis is defined to be a subset of the topology), and therefore $U \cap V$ will be in the topology. So $U \cap V$ will also have to equal the union of some collection of sets in $\mathcal{B}$, $\{B_\alpha\}_{\alpha\in\lambda}$. Therefore if $p \in U \cap V$, $p \in \cup_{\alpha\in\lambda} B_\alpha$, and therefore $p \in B_\alpha \subseteq \cup_{\alpha\in\lambda} B_\alpha = U \cap V$ for some $\alpha \in \lambda$.

($\Leftarrow$) Suppose 1-3 above hold for a set $\mathcal{B}$. Let $\mathcal{T}$ be the set of all possible unions of elements in $\mathcal{B}$ together with the individual elements in $\mathcal{B}$. We want to show that $\mathcal{T}$ is a topology.

1. $\underline{\emptyset \in \mathcal{T}}$. Assumption (1) gives us $\emptyset \in \mathcal{B}$, so $\emptyset \in \mathcal{T}$.
2. $\underline{X \in \mathcal{T}}$. Assumption (2) gives us that for each $p \in X$, there is a set $U_p \in \mathcal{B}$ containing p. So we can build a union of sets in $\mathcal{B}$, $\cup_{p\in X} U_p = X$, since each $p \in X$ will be in U_p and all of the U_p's will be subsets of X. We have built our topology $\mathcal{T}$ in such a way as to guarantee that this union equal to X will be one of its elements.

3. $\underline{A \in \mathcal{T} \text{ and } B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}}$. Suppose $A \in \mathcal{T}$ and $B \in \mathcal{T}$. If $A \cap B = \emptyset$, then $A \cap B \in \mathcal{T}$. Suppose $A \cap B$ is nonempty. Let $p \in A \cap B$. We can write A and B as unions of sets in $\mathcal{B}$: $A = \cup_{\alpha \in \lambda} B_\alpha$ and $B = \cup_{\beta \in \gamma} B_\gamma$. So $p \in B_\alpha$ and $p \in B_\gamma$ for some B_α, $B_\gamma \in \mathcal{B}$. Since p is a point in the intersection of two sets in $\mathcal{B}$, Assumption (3) gives us that there exists a set W in $\mathcal{B}$ such that $p \in W \subseteq (B_\alpha \cap B_\gamma)$.
   We want to show that $W \subseteq A \cap B$. So, let $w \in W$. Then $w \in B_\alpha$ and $w \in B_\gamma$, so w is in the union of the B_α's and the union of the B_γ's, and thus $w \in A$ and $w \in B$. So $w \in A \cap B$. This means that for each $p \in A \cap B$, we can find $W_p \in \mathcal{B}$ such that $p \in W_p \subseteq A \cap B$. It follows that $A \cap B = \cup_{p \in A \cap B} W_p$ and that this collection of unions of sets in $\mathcal{B}$ must be in $\mathcal{T}$. So $A \cap B \in \mathcal{T}$.

4. $\underline{\text{If } \{A_\alpha\}_{\alpha \in \lambda} \text{ is any collection of sets each of which is in } \mathcal{T}, \text{ then } \cup_{\alpha \in \lambda} A_\alpha \in \mathcal{T}}$. We get this from our definition of $\mathcal{T}$. Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of sets each of which is in $\mathcal{T}$. So for all $\alpha \in \lambda$, $A_\alpha = \cup_{\beta \in \gamma} B_\beta$ where the B_β's are all in $\mathcal{B}$. Because a union of unions of sets in $\mathcal{B}$ will be a union of sets in $\mathcal{B}$, we will have $\cup_{\alpha \in \lambda} A_\alpha \in \mathcal{T}$.

∎

Now that we know how to recognize a basis for a topology, let $X = \{a, b, c\}$. We want to build two bases, $\mathcal{B}_1$ and $\mathcal{B}_2$, that will give rise to the same topology. Let $\mathcal{B}_1 = \{\emptyset, \{a\}, \{b\}, \{c\}\}$. We see this satisfies the requirements of Thm 3.3 above. When we take all possible unions, we get $\mathcal{T} = \mathcal{P}(X)$. Now, let $\mathcal{B}_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. This too will expand to give us $\mathcal{T} = \mathcal{P}(X)$. So we see that there can be multiple bases for a given topology.

Now that we know about bases, we are ready to begin our discussion of metrizability. Recall the three basic requirements of distance formula.

Definition 3.4 Let X be a set. A function d from $X \times X$ into $\mathbb{R}^+ \cup \{0\}$ is a **metric** if and only if the following hold:

1. $d(x, y) = 0$ if and only if $x = y$,

2. $d(x, y) = d(y, x)$ for all x, $y \in X$, and

3. $d(x, z) \leq d(x, y) + d(y, z)$.

So, there are many different ways to define a metric on a given set X. We get metrizability on a set $(X, \mathcal{T}$ when the $\mathcal{T}$ coincides some possible metric d on X. More rigorously:

Definition 3.5 Suppose X is a set, d is a metric for X, $p \in X$, and $\epsilon > 0$. Then the **open ball** of radius ϵ about p is defined by $B_\epsilon(p) = \{x \in X : d(x,p) < \epsilon\}$. The d-**metric topology** for X is the topology whose basis consists of $B_\epsilon(p)$ for all $\epsilon > 0$ and all $p \in X$.

Definition 3.6 Suppose $(X, \mathcal{T})$ is a topological space. Then $(X, \mathcal{T})$ is a **metric space**, or **metrizable**, if and only if there is a metric d on X for which $\mathcal{T}$ is the d-metric topology. The metric space is denoted (X, d).

Notice that this definition corresponds directly to the usual topology on $\mathbb{R}^n$ mentioned earlier. There are other, less intuitive examples of metrizable topological spaces. Consider a set X under the trivial metric:

$$d(x,y) = \begin{cases} 0 & : x = y \\ 1 & : x \neq y \end{cases}$$

When $\epsilon \leq 1$, for all $p \in X$ $\mathrm{B}_\epsilon(p)$ will contain only p. On the other hand, when $\epsilon \geq 1$, $\mathrm{B}_\epsilon(p) = X$ for all $p \in X$. Therefore, the basis consists of all singleton sets as well as X itself, giving rise to a topology that contains all possible unions of elements in X – precisely the discrete topology. So, we see that a set under the discrete topology is always metrizable by way of the trivial metric.

Now that we have a formal definition of a metric space, let's return to our earlier definitions of open and closed sets in the context of metric spaces. We can see how definition of an open set in a metric space arises from previous material:

Definition 3.7 A subset U of a metric space X is an open set provided that it is the union of a collection of open balls.

By the topological definition of openness a set is open if and only if it is in the topology. By the definition of a basis, a set is in the topology if and only if it is the union of elements of the basis. And, by the definition of metrizability, in a metric space, the basis is made up of open balls. So we get that all open sets are unions of open balls and all unions of open balls are open sets.

The metric space definition of a closed set is no different from our topological definition – a set containing all of its limit points. However, the metric space definition of limit point *does* differ from the general topological definition:

Definition 3.8 Let X be a metric space, let S be any subset of X, and let $x \in X$. We say that x is a limit point of S if it satisfies any one of the following[1]:

1. There exists a sequence of points in $S \setminus \{x\}$ converging to x.
2. There exists a sequence of distinct points of S converging to x.
3. For all $r > 0$, $\mathrm{B}_r(x)$ contains infinitely many points of S.
4. For all $r > 0$, $\mathrm{B}_r(x)$ contains a point of $S \setminus \{x\}$.

We recognize (4) as equivalent to our topological definition. We will not spend too much time looking at (2) and (3), because they require S, and therefore X to be infinite. What we can take from this is that, while metric spaces can be finite, there are no limit points in a finite metric space. Since there can be limit points in a finite topological space, these definitions are not particularly useful for our purposes. Instead, the alternate definition we want to consider is condition (1). In order to examine this in the context of a general topological space, we will need the topological definition of convergence:

Definition 3.9 Let $P = \{p_i\}_{i \in \mathbb{N}}$ be a sequence of points in a space X. Then the sequence P **converges to** a point x if and only if for every open set U containing x, there is an integer M so that for each $m > M$, $p_m \in U$.

In Real Analysis, we often use the sequence definition of limit point. However, in a general topological space (1) and (4) are not equivalent – it turns out that (1) is the stronger condition.

Theorem 3.10 Let X be a topological space, let S be a subset of X, and let $x \in X$. If there exists a sequence of points in $S \setminus \{x\}$ converging to x, then x is a limit point for S.

Proof:
Let (s_n) be a sequence in $S \setminus \{x\}$ converging to x. Let U be an open set containing x. Then there exists $M \in \mathbb{N}$ such that for all $m > M$, $s_m \in U$. Since (s_n) is in $S \setminus \{x\}$, we can certainly choose $p > M$ such that $s_p \neq x$. So $s_p \in U \setminus \{x\} \cap S$. In this way, for all open sets U, we can find an element in

[1]We will use some of these equivalent conditions later on, so we will prove the ones we need further on.

$U \setminus \{x\} \cap S$, which allows us to conclude that x is a limit point for S.
∎

The converse of this statement does not hold. Consider the following example:

Let $X = [0, 1]$ and define a topology on X by declaring the open sets to be $\emptyset$ and those subsets of X whose complement is at most countable. Consider the set $A = [0, 1)$. Clearly, any open set containing 1 must contain a point of $[0, 1)$, so 1 is a limit point for $[0, 1)$. Now, let (s_n) be a sequence of points in $[0, 1)$. Then, since $\{s_n : n \in \mathbb{N}\}$ is a countable set, its complement will be an open set under the given topology. Since $1 \notin [0, 1)$, for all $n \in \mathbb{N}$, $s_n \neq 1$. Therefore, $\{s_n : n \in \mathbb{N}\}^{\mathcal{C}}$ is an open set containing 1 that does not contain any points of the sequence (s_n). So (s_n) does not converge to 1.

This shows us that sequences and limit points in topological spaces may behave differently from those in metric spaces. Consider another characteristic of convergent sequences in metric spaces:

Theorem 3.11 A convergent sequence s_n in a metric space X can only have one limit.

Proof:

Let s_n be a convergent sequence in a metric space (X, d). Suppose (s_n) converges to a point $x \in X$. Suppose $y \neq x$. So $d(x, y) = a$ for some $a > 0$. Consider the open set $\mathrm{B}_{\frac{a}{2}}(y)$. Let $M \in \mathbb{N}$. Since s_n converges to x there exists $N \in \mathbb{N}$ such that for all $n > N$, $s_n \in \mathrm{B}_{\frac{a}{2}}(x)$. Let $n > M$ such that $n > N$ as well. Since $n > N$, we know $s_n \in \mathrm{B}_{\frac{a}{2}}(x)$. But $\mathrm{B}_{\frac{a}{2}}(x) \cap \mathrm{B}_{\frac{a}{2}}(y) = \emptyset$. So $s_n \notin \mathrm{B}_{\frac{a}{2}}(y)$. Therefore, s_n must not converge to y. So we can conclude that for all $z \in X$, if $z \neq x$, s_n does not converge to z. So x is the unique limit of s_n. ∎

In a general topological space, on the other hand, a convergent sequence need not have a unique limit. Consider the space $X = \{1, 2, 3\}$ under the indiscrete topology. By our definition, the sequence $s_n = 1, 2, 1, 2, 1, 2...$ converges to 1 in X: X is the only open set containing 1, and for all $n \in \mathbb{N}$, $s_n \in X$. By the same logic, (s_n) also converges to 2, and even 3. This example reveals another unexpected property of general topological spaces – oscillating sequences can converge.

We will need to use several different definitions of limit points in metric spaces in the proofs ahead. In order to be rigorous in our examination, we will go ahead and prove a variation of the metric space definition of limit points mentioned above.

Theorem 3.12 Let (X, d) be a metric space, let $S \subseteq X$, and let $x \in X$. The following are equivalent.

- x is a limit point of S.
- There exists a sequence of distinct terms in S converging to x.
- For all $\epsilon > 0$ there exists $y \in X$ such that $0 < d(x, y) < \epsilon$.

Proof:
($1 \Rightarrow 2$) Let x be a limit point of S. So for all open sets U, $(U \setminus \{x\}) \cap S$ is nonempty. We can use this to choose a sequence (s_n) in S such that $x \in (\mathrm{B}_{\frac{1}{n}} \setminus \{x\} \cap S)$. Now, let V be an open subset of X containing x. Then we can find $\epsilon > 0$ such that $\mathrm{B}_\epsilon(x) \subseteq V$. Let $M \in \mathbb{N}$ such that $\frac{1}{M} < \epsilon$. Let $m > M$. Then, we can see from our definition of (s_n) that $d(s_m) < \frac{1}{m} < \frac{1}{M} < \epsilon$. So $s_m \in \mathrm{B}_\epsilon(x) \subseteq V$. Therefore, (s_n) converges to x.
Now, we need to extract a subsequence (s_{n_i}) of distinct terms from (s_n). We can do this by induction:
Base Case: Let $n_1 = 1$.
Induction Hypothesis: Suppose we've chosen $n_1 < n_2 < ... < n_k$ for some $k \geq 1$ such that $d(s_{n_1}, x) > d(s_{n_2}, x) > ... > d(s_{n_k}, x)$.
Induction Step: Find $r > 0$ such that $r < d(s_{n_k}, x)$. (Note, we chose the s_n's in such a way that none equaled x, so we can be sure this distance will be positive.) Choose $q \in \mathbb{N}$ such that $q > n_k$ and $\frac{1}{q} < r$. By our definition of the s_n's, $d(s_q, x) < \frac{1}{q} < r < d(s_{n_k}, x)$. Let $n_{k+1} = q$. So we get $n_{k+1} > n_k$ and $d(s_{n_{k+1}}, x) < d(s_{n_k}, x)$. Now we have a sequence n_i such that for i, $j \in \mathbb{N}$, $d(s_{n_i}, x) = d(s_{n_j}, x)$ if and only if $i = j$. The well-definedness of d allows us to conclude that $s_{n_i} = s_{n_j}$ if and only if $i = j$. So we have created a sequence of distinct terms, (s_{n_i}). Since this is a subsequence of (s_n) and (s_n) converges to x, (s_{n_i}) converges to x.
($2 \Rightarrow 3$) Suppose there exists a sequence (s_n) of distinct points in S that converges to x. Let $\epsilon > 0$. $\mathrm{B}_\epsilon(x)$ is an open set containing x, so by the definition of convergence, there exists $N \in \mathbb{N}$ such that for all $n > N$, $s_n \in \mathrm{B}_\epsilon(x)$. Since (s_n) is a sequence of distinct terms, we can certainly find $n > N$ such that $s_n \neq x$. So $s_n \in S$ and $0 < d(s_n, x) < \epsilon$.
($3 \Rightarrow 1$) Suppose that for every $\epsilon > 0$ there exists a point y of S with $0 < d(x, y) < \epsilon$. Let U be an open set in X containing x. Then, since X is a metric space, for some $r > 0$, $\mathrm{B}_r(x) \subseteq U$. By our assumption, there will exist a point $y \in S$ such that $0 < d(x, y) < r$. So $y \in \mathrm{B}_r(x)$ and $y \neq x$. So $y \in (U \setminus \{x\}) \cap M$. Since U was arbitrary, this intersection is nonempty for all open sets $U \in X$. So x is a limit point for M. ∎

Related to the above theorem, there is an equivalent definition of what it means for a topology to be the d-metric topology for a given distance function d.[2] We will use this definition in our final metrization theorem, so we will prove it now.

Theorem 3.13 Let $(X, \mathcal{T})$ be a topological space, and let d be a metric on X. $\mathcal{T}$ is the d-metric topology on X if and only if the following statement holds:

> x is a limit point of S in $(X, \mathcal{T})$ if and only if for all $\epsilon > 0$ there exists $y \in S$ such that $0 < d(x, y) < \epsilon$.

Proof:

Let $(X, \mathcal{T})$ be a topological space, and let d be a metric on X. The forward direction of the implication is that, if $\mathcal{T}$ is the d-metric topology on X, the above statement holds. We have already proven this in Theorem 3.12 above, in showing that Statements 1 and 3 were equivalent. Thus, we need only prove the reverse direction.

So, suppose that x is a limit point of S in $(X, \mathcal{T})$ if and only if for all $\epsilon > 0$ there exists $y \in S$ such that $0 < d(x, y) < \epsilon$. We want to show that this means that $\mathcal{T}$ is the d-metric topology on X, or, equivalently, that for all $U \in \mathcal{T}$, U is the union of open balls. So, let $U \in \mathcal{T}$. So U is open. Let $p \in U$. Suppose that for all $\epsilon > 0$, $\mathrm{B}_\epsilon(p) \cap U^{\mathcal{C}} \neq \emptyset$. Then, since $p \in U$, for all $\epsilon > 0$ there exists $y \in U^{\mathcal{C}}$ such that $0 < d(p, y) < \epsilon$. This means that p is a limit point of $U^{\mathcal{C}}$. But since U is open, $U^{\mathcal{C}}$ is closed, and must contain all of its limit points. So $p \in U^{\mathcal{C}}$. Since p cannot be in both a set and its complement, this is a contradiction. So there must exist some $\epsilon_p > 0$ such that $\mathrm{B}_{\epsilon_p}(p) \cap U^{\mathcal{C}} = \emptyset$. So $\mathrm{B}_{\epsilon_p}(p) \subseteq U$. Thus, for all $p \in U$, there exists $\epsilon > 0$ such that $\mathrm{B}_{\epsilon_p}(p) \subseteq U$. So U is a union of open balls.

∎

4 Separation Properties

In a metric space, if we have two distinct points x and y, we know we can create open balls $\mathrm{B}_{\frac{d(x,y)}{2}}(x)$ and $\mathrm{B}_{\frac{d(x,y)}{2}}(y)$ so that the two balls do not intersect. In essence, we have *separated* x and y by containing them in non-intersecting open sets. Now, what if we wanted to separate a point from a closed set? Or two closed sets from each other? As we will prove momentarily, all of

[2]This theorem comes from [W-D], p.22.

these types of separation will be possible in a metric space. However, these properties are not guaranteed in a general topological space. Therefore, taking a closer look at the separation properties will help us understand the characteristics that set metric spaces apart from general topological spaces.

Definition 4.1 Let $(X, \mathcal{T})$ be a topological space.

1. X is $\mathbf{T}_1$ if and only if every singleton set in X is a closed set.

2. X is **Hausdorff** if and only if for each pair of points x, $y \in X$, there are disjoint open sets U and V in $\mathcal{T}$ such that $x \in U$ and $y \in V$.

3. X is **regular** if and only if for each $x \in X$ and closed set $A \in X$, with $x \notin A$, there exist disjoint open sets U and V such that $x \in U$ and $A \subseteq V$.

4. X is **normal** if and only if for each pair of disjoint closed sets A and $B \in X$, there exist dijoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

As mentioned above, metric spaces possess all of these four properties. However, before we prove this formally, we may gain some insight by looking at topological spaces in which these properties do not hold.

Let $\mathcal{T}$ be a topology defined on $\mathbb{R}^2$ such that the open sets are the empty set and all open circles around $(0, 0)$ with positive even integer radii.[3] This topological space is not T_1, Hausdorff, regular, or normal, as the following discussion and the figures below will demonstrate.

$\mathbf{T}_1$ In order for a singleton set $\{x\}$ to be not closed, it must have limit points other than x. So, if we have a space where every open set containing x also contains y, x will necessarily be a limit point of $\{y\}$ not contained in that set, and the space will fail to be T_1. For example, in our topological space $(\mathbb{R}^2, \mathcal{T})$, $\{(0, 0)\}$ will not be closed, since all elements of $\mathrm{B}_2((0, 0))$ under the Euclidean metric will be limit points for the origin.

[3]Note: intuitively, this corresponds to $\{\emptyset\} \cup \{B_r((0, 0)) : r = 2n \text{ for some } n \in \mathbb{N}\}$. However, because the definition of these open balls depends on the usual metric on $\mathbb{R}^2$, and this particular space is not metrizable, we will refer to open and closed *circles* instead of balls, to avoid confusion.

Hausdorff What allows us to separate two points x and y is the existence of two disjoint open sets, one containing x and one containing y. $(\mathbb{R}^2, \mathcal{T})$ is a non-Hausdorff space because we will never be able to separate $(1, 0)$ and $(-1, 0)$ with disjoint open sets.

Hausdorff is actually a stronger condition than T_1; that is, all Hausdorff spaces are T_1 but not all T_1 spaces are Hausdorff. Consider a space X containing distinct points x and y. If the space is Hausdorff, then we can find open balls U and V containing x and y respectively, so that there's an open set U where $U \setminus \{y\} \cap \{x\} = \emptyset$. Therefore y cannot be a limit point for x. We can do this for any point in the space, so $\{x\}$ can have no limit points other than x, making it a closed set. So the space must be T_1. However, if the space is T_1 all we know is that there is an open set around y not containing x and an open set around x not containing y. We cannot be certain that these sets are disjoint, and therefore we do not get the Hausdorff property.

An example of a space that is T_1 but not Hausdorff is $\mathbb{R}$ under the finite complement topology mentioned at the start of the paper. To see this, let $\{x\}$ be a singleton set in $\mathbb{R}$. Then $\{x\}^{\mathcal{C}}$ certainly has a finite complement, and is therefore open. So $(\{x\}^{\mathcal{C}})^{\mathcal{C}} = \{x\}$ is closed. This means that this space is T_1. However, suppose that x, $y \in \mathbb{R}$ with $x \neq y$. Suppose U is an open set containing x. In order for the space to be Hausdorff, we would have to find an open set $V \in U^{\mathcal{C}}$ such that $y \in V$. But since U is open, $U^{\mathcal{C}}$ must be finite. Thus, any subset of $U^{\mathcal{C}}$ will also be finite, meaning its complement will not be finite. So there are no open subsets of $U^{\mathcal{C}}$, and the space is not Hausdorff.

Regular In order to separate a point x from a closed set M, we need disjoint open sets, one containing x, the other containing M. Notice that this is like Hausdorff, but slightly harder to satisfy. Suppose we were trying to prove that a Hausdorff set is regular. (This result is false in general!) We could find pairs of disjoint open sets for every combination of x and a point $m \in M$. The union of the open sets around points in M would form an open set. However, in order to maintain the original disjointness, we'd have to take the intersection of all of the open sets containing x. Since this is not necessarily a finite intersection, we do not know that the intersection will be open. So we are stuck. If we are certain that our space is finite, then M will be finite, and so we will be able to prove that Hausdorff implies regular.

In terms of the converse, while regularity alone does not imply that a space is Hausdorff, a space that is regular and T_1 will certainly be

Hausdorff. In this case, we can view two points x and y as a point and a closed sets x and $\{y\}$ and then use regularity to find disjoint open sets, one containing each.

Let's return to our topology on the Euclidean Plane. Consider the closed set $\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 4\}$. (This is the intersection of two closed sets in $(\mathbb{R}^2, \mathcal{T})$: the closure of the circle of radius 4 and its complement.) All open sets containing this closed circle must contain the open set directly inside – that is $\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 4\}$. So a point like $(3, 0)$ that is in the inner open set and not the closed circle can never be separated from the closed circle.

Normal Normal spaces are the spaces we will be most concerned with in this paper. Normality is, in some ways, at the the top of the separation properties hierarchy. Given a normal, T_1 space, we get the rest of the properties for free: normal and T_1 implies regular[4], and regular and T_1 implies Hausdorff. In fact, some definitions of normality include the T_1 property.

In our topological space $(\mathbb{R}^2, \mathcal{T})$, we cannot separate the closed set $A = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 4\}$ mentioned under regularity with the closed set $B = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \leq 2\}$. All open sets containing A must contain $\{(x, y) \in \mathbb{R}^2 : 2 < \sqrt{x^2 + y^2} < 8\}$ and all open sets containing B must contain $\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 4\}$, creating an inevitable overlap of the open set $\{(x, y) \in \mathbb{R}^2 : 2 < \sqrt{x^2 + y^2} < 4\}$. So $(\mathbb{R}^2, \mathcal{T})$ fails to be normal.

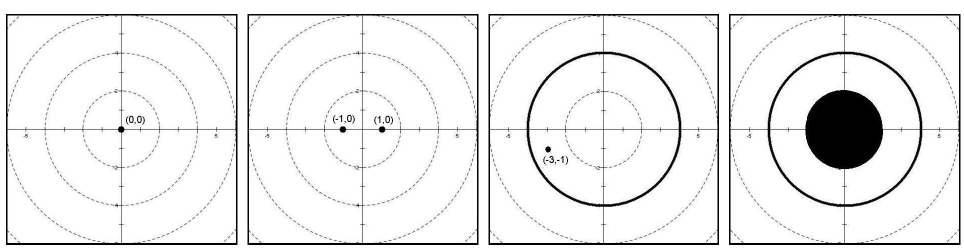

Figure 1: a) Not T_1 b) Not Hausdorff c) Not regular d) Not normal

Now, let's look at these properties in terms of metric spaces. As we saw at the beginning of the section, all metric spaces must be Hausdorff. Therefore,

[4] A proof which runs similar to the proof of regular and T_1 implies Hausdorff, and which we will prove below in Corollary 4.5.

they must all be T_1 as well. To demonstrate regularity and normality in a metric space, we will need a few intermediate results.

Lemma 4.2 A topological space X is regular if and only if for each point p in X and open set U containing p, there is an open set V such that $p \in V$ and $\bar{V} \subseteq U$.

Proof:
($\Rightarrow$) Let $p \in X$ and let U be an open set containing p. By Thm 2.6, $U^{\mathcal{C}}$ will be closed. Since $p \in U$, $p \notin U^{\mathcal{C}}$. Since X is regular, we can find open sets M and N such that $p \in M$, $U^{\mathcal{C}} \subseteq N$, and $M \cap N = \emptyset$. It remains to show that $\bar{M} \subseteq U$. So, let $m \in \bar{M}$.

- Suppose $m \in M$. Then $m \notin N$. So $m \notin U^{\mathcal{C}}$. Thus $m \in U$.
- Suppose $m \notin M$. Then m is a limit point for M. Since $N \cap M = \emptyset$, clearly $(M \setminus \{m\} \cap N = \emptyset$. Because N is an open set and m is a limit point of M, this means that N cannot contain m. So, since $U^{\mathcal{C}} \subseteq N$, $m \notin U^{\mathcal{C}}$. Therefore, $m \in U$.

Since in either case, we get $m \in U$, we may conclude that $\bar{M} \subseteq U$.
($\Leftarrow$) Suppose that for each point p in X and open set U containing p, there is an open set V such that $p \in V$ and $\bar{V} \subseteq U$. Let $x \in X$ and let A be a closed subset of X such that $x \notin A$. Then $A^{\mathcal{C}}$ is an open set that contains x. So there exists an open set U such that $x \in U$ and $\bar{U} \subseteq A^{\mathcal{C}}$. Let $V = \bar{U}^{\mathcal{C}}$. So V is open. Now, let $a \in A$. Then $a \notin A^{\mathcal{C}}$, so $a \notin \bar{U}$. Thus $a \in \bar{U}^{\mathcal{C}} = V$. So we have open sets U and V such that $x \in U$ and $A \subseteq V$. Furthermore, we can tell by our definition of V that $U \cap V = \emptyset$. ∎

Lemma 4.3 A topological space X is normal if and only if for each closed set A in X and open set U containing A, there is an open set V such that $A \subseteq V$ and $\bar{V} \subseteq U$.

Proof:
($\Rightarrow$) Let A be a closed set in a normal space X and let U be an open set containing A. So by Thm 2.6, $U^{\mathcal{C}}$ will be a closed set. We know that if $a \in A$, since $A \subseteq U$, $a \in U$, so $a \notin U^{\mathcal{C}}$. So A and $U^{\mathcal{C}}$ are disjoint. Now, we can use the definition of a normal space to find open sets M and N such that $A \subseteq M$, $U^{\mathcal{C}} \subseteq N$, and $M \cap N = \emptyset$. It remains to show that $\bar{M} \subseteq U$. The argument follows the same as in the previous proof, so it will be omitted here. We may conclude that $\bar{M} \subseteq U$.

($\Leftarrow$) Suppose that for each closed set A in X and open set U containing A, there is an open set V such that $A \subseteq V$ and $\bar{V} \subseteq U$. Let A and B be disjoint closed subsets of X. Then $B^{\mathcal{C}}$ is an open set that contains A. So there exists an open set U such that $A \subseteq U$ and $\bar{U} \subseteq B^{\mathcal{C}}$. Let $V = \bar{U}^{\mathcal{C}}$. So V is open. Now, let $b \in B$. Then $b \notin B^{\mathcal{C}}$, so $b \notin \bar{U}$. Thus $b \in \bar{U}^{\mathcal{C}} = V$. So we have open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Furthermore, we can tell by our definition of V that $U \cap V = \emptyset$. ▮

Theorem 4.4 All metric spaces are normal.

Proof:
Let X be a metric space.
Let A and B be disjoint closed sets in X. So $B^{\mathcal{C}}$ is open and contains A. Therefore, since X is a metric space, for all $a \in A$, we can find $\mathrm{B}_{r_a}(a) \subseteq B^{\mathcal{C}}$. Similarly, we want to find $\mathrm{B}_{r_b}(b) \subseteq A^{\mathcal{C}}$ for all $b \in B$.
Let $U = \cup_{a \in A} \mathrm{B}_{\frac{r_a}{2}}(a)$ and $V = \cup_{b \in B} \mathrm{B}_{\frac{r_b}{2}}(b)$. As unions of open sets, U and V are open. We can see from our construction of U and V that $A \subseteq U$ and $B \subseteq V$. It remains to show that U and V are disjoint.
We will proceed by contradiction. Suppose that there exists some $x \in U \cap V$. So $x \in \cup_{a \in A} \mathrm{B}_{\frac{r_a}{2}}(a)$ and $x \in \cup_{b \in B} \mathrm{B}_{\frac{r_b}{2}}(b)$. This means that for some $a \in A$, $d(x, a) < \frac{r_a}{2}$ and for some $b \in B$, $d(x, b) < \frac{r_b}{2}$. We can assume, without loss of generality, that $r_a \geq r_b$. Using the triangle inequality, we see that

$$\begin{aligned} d(a,b) &\leq d(a,x) + d(x,b) \\ &< \frac{r_a}{2} + \frac{r_b}{2} \\ &\leq \frac{r_a}{2} + \frac{r_a}{2} = r_a \end{aligned}$$

So $b \in \mathrm{B}_{r_a}(a)$. But this open ball was chosen specifically to exclude all elements of B. So this is not possible. Therefore, we conclude that $U \cap V = \emptyset$.
▮

So we see that all metric spaces are normal. From this, regularity of metric spaces follows nicely.

Corollary 4.5 All metric spaces are regular.

Proof:
Let X be a metric space. Let $p \in X$ and let A be a closed set in X such that $p \notin A$. Because metric spaces are all T_1, $\{p\}$ will be a closed set. Therefore,

since Theorem 4.4 guarantees that X is normal, we can find disjoint open sets U and V such that $\{p\} \subseteq U$ (i.e. $p \in U$) and $A \subseteq V$. So X is regular. ∎

Since a singleton set is closed (by the T_1 property of metric spaces), we get regularity for free from the combination of normality and T_1.

So, we know that any set that is not T_1, Hausdorff, regular, and normal cannot be a metric space. However, there are sets with all four of these characteristics that are still not metrizable. Consider, for example, the right, half-open interval topology on $\mathbb{R}$. [5] Define a topology on $\mathbb{R}$ that has the basis of all sets of the form $[a, b)$ for a, $b \in X$.

We can see that this space is T_1: Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$ such that $y \neq x$. Suppose $y < x$. Then there exists z such that $y < z < x$. So $[y, z)$ is an open set containing y that does not intersect $\{x\}$. Similarly, if $y > x$, there exists z such that $x < y < z$. So $[y, z)$ is still an open set containing y that does not intersect $\{x\}$. Therefore, if $y \neq x$, y cannot be a limit point for $\{x\}$. So $\{x\}$ is closed.

We can also demonstrate that $\mathbb{R}$ under the half-open topology is normal. Let A and B be disjoint, closed sets in this space. Then, since $B^{\mathcal{C}}$ is an open set containing A, for each $a \in A$, there exists a basis element $[a, x_a)$ such that $[a, x_a) \subseteq B^{\mathcal{C}}$). Define $U = \cup_{a \in A}[a, x_a)$, and using a parallel method, define $V = \cup_{b \in B}[b, x_b)$. As unions of open sets, U and V will both be open. Furthermore, by definition, $A \subseteq U$ and $B \subseteq V$. Now, suppose there existed $z \in U \cap V$. Then for some $a^* \in A$ and some $b^* \in B$, $z \in [a, x_a)$ and $z \in [b, x_b)$. Suppose without loss of generality that $a < b$. Then $z \geq b > a$. In order for z to be in $[a, x_a)$, $x_a > z$. Therefore, $x_a > z \geq b > a$. So $b \in [a, x_a)$. But this interval was chosen specifically to exclude all elements of B. So this cannot be. Therefore, we must have $U \cap V = \emptyset$. So we may conclude that our space is normal.

Normality and T_1 give us regularity and the Hausdorff property, as mentioned above. So this space has all four of the separation properties. Yet it is not metrizable! The proof of this fact is beyond the scope of this paper, but it can be found in Steen and Seebach's *Counterexamples in Topology.*

So, since the four separation properties are not enough to guarantee metrizability, it remains to determine what further qualifications we must add to the separation properties to ensure that a given space will be metrizable.

Before we move on to the next section, we will prove a brief lemma that relates to separation properties, and that we will need in our final metrization theorem.

[5] This example, as well as the following proof of normality, come from [S-S], p.75.

Lemma 4.6 Let $(X, \mathcal{T})$ be a T_1, normal topological space with a basis $\mathcal{B}$. If $x \in X$ and M is a closed subset of X such that $x \notin M$, then there exist basis elements B^* and B° such that $x \in B^* \subseteq \bar{B^*} \subseteq B^\circ \subseteq M^{\mathcal{C}}$.

Proof:

Let $x \in X$ and let M be a closed subset of X such that $x \notin M$. Then, by Theorem 3.3 (2), since $x \in M^{\mathcal{C}}$ which is an open set, there exists an element of the basis, B°, such that $x \in B^\circ \subseteq M^{\mathcal{C}}$. Now, since (B°) is an open set containing $\{x\}$, we can use normality (and the fact that, since X is T_1, $\{x\}$ is a closed set) to find an open set U such that $x \in U \subseteq \bar{U} \subseteq B^\circ$. Now, we return to Theorem 3.3 (2) to find a basis element B^* such that $x \in B^* \subseteq U$. Since $B^* \subseteq U$, Theorem 2.7 gives us that $\bar{B^*} \subseteq \bar{U}$. So we get $x \in B^* \subseteq \bar{B^*} \subseteq \bar{U} \subseteq B^\circ \subseteq M^{\mathcal{C}}$. If we ignore the $\bar{U}$, we see that the above statement is exactly what we wanted to prove. ∎

5 Countability, Covers, and Continuity

As we saw previously, a given topology can have many different bases. Since the basis generates the topology, topologies are sometimes classified by whether we can find a certain type of basis for it. One such property has to do with countability. There are several different types of countability, but the one required in the metrization theorem we are working towards is second countability.

Definition 5.1 A space X is **second countable** provided that X has a countable basis.

The Euclidean Plane, for example, is second countable with its countable basis being the open balls of rational radii centered at points with rational coordinates. An example of a space that is not second countable would be any uncountable set under the discrete topology, since any basis would have to include all uncountably many of its singleton open sets.

Along with bases, we will want to look at open covers of topological spaces.

Definition 5.2 Let A be a subset of X and let $\mathcal{B} = \{B_\alpha\}_{\alpha \in \lambda}$ be a collection of subsets of X. Then $\mathcal{B}$ is a **cover** of A if and only if $A \subseteq \cup_{\alpha \in \lambda} B_\alpha$. $\mathcal{B}$ is an **open cover** if and only if each B_α is open.

In particular, we will be interested in covers with countable subcovers.

Definition 5.3 A space X is **Lindelöf** provided that every open cover of X has a countable subcover.

As we might expect, since the open sets in open covers come from the topology which in turn comes from the basis, all second countable spaces will necessarily be Lindelöf as well.

Theorem 5.4 Every second countable space is Lindelöf.

Proof:

Let X be a second countable space with basis $B_1, B_2, ...$ Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. Let $p \in X$. Then by Theorem 3.3 (2), there exists B_{i_p} in the basis such that $p \in B_{i_p} \subseteq U_{\alpha^*}$. We will repeat this process for all elements in X. Let $C = \{i \in \mathbb{N} : i = i_p \text{ for some } p \in X\}$. Now, consider the set $V = \{B_c | c \in C\}$. We know that since there are only countably many B_i's, there can only be countably many elements in V. Also, by construction, for all $c \in C$, $B_c \subseteq U_\alpha$ for some $\alpha \in \Lambda$. Therefore, we can build a collection of countably many sets: for each $B_c \in V$, choose $U_c \in \{U_\alpha\}_{\alpha\in\Lambda}$ such that $B_c \subseteq U_c$. We propose $\{U_c\}_{c\in C}$ as our countable subcover.

To demonstrate this, let $x \in X$. Then $x \in B_c$ for some $c \in C$. Therefore, $x \in U_c$. So $x \in \cup_{c\in C} U_c$. Therefore, $\{U_c\}_{c\in C}$ is a cover for X, and since it is comprised only of U_α's from the original cover, it is a countable subcover of $\{U_\alpha\}_{\alpha\in\Lambda}$. ∎

Theorem 5.5 Every open cover of a closed set in a Lindelöf space has a countable subcover.

Proof:

Let A be a closed subset in a Lindelöf space, X. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover of A. By Thm 2.6, $A^{\mathcal{C}}$ is open. We know that for all $x \in X$, if $x \in A$, $x \in \{U_\alpha\}_{\alpha\in\lambda}$ and if $x \notin A$, $x \in A^{\mathcal{C}}$. So $\{U_\alpha\}_{\alpha\in\lambda} \cup \{A^{\mathcal{C}}\}$ forms an open cover of X. Therefore, by the definition of Lindelöf, we can extract a countable subcover $\{V_i\}_{i\in\mathbb{N}}$.

The only set that might keep $\{V_i\}_{i\in\mathbb{N}}$ from being a subcover of $\{U_\alpha\}_{\alpha\in\lambda}$ is $A^{\mathcal{C}}$. So, we propose $\{V_i\}_{i\in\mathbb{N}} \setminus \{A^{\mathcal{C}}\}$ as our countable subcover. We can be certain that all elements of $\{V_i\}_{i\in\mathbb{N}} \setminus \{A^{\mathcal{C}}\}$ come from $\{U_\alpha\}_{\alpha\in\lambda}$, so it remains to show that $A \subseteq \cup_{i\in\mathbb{N}} U_i$. Let $a \in A$. So, since $A \subseteq X$, $a \in X$. So $a \in \cup_{i\in\mathbb{N}} V_i$. Furthermore, since $a \in A$, $a \notin A^{\mathcal{C}}$. So $a \in \cup_{i\in\mathbb{N}} V_i \setminus A^{\mathcal{C}}$. Thus, we have our countable subcover. ∎

The Lindelöf property happens to transform a regular space into a normal space. To prove this, though, we will first need the following lemma.

Theorem 5.6 (**Normality Lemma**) Let A and B be subsets of a topological space X and let $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be two collections of open sets such that

1. $A \subseteq \cup_{i\in\mathbb{N}} U_i$,
2. $B \subseteq \cup_{i\in\mathbb{N}} V_i$
3. for each i in $\mathbb{N}$, $\bar{U_i} \cap B = \emptyset$ and $\bar{V_i} \cap A = \emptyset$.

Then there are open sets U and V so that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

Proof:

Let X, A, B, $\{U_i\}_{i\in\mathbb{N}}$, and $\{V_i\}_{i\in\mathbb{N}}$ as above. Define $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ such that $A_1 = U_1$, $A_n = U_n \cap (\cup_{i<n}\bar{V_i})^{\mathcal{C}}$ and $B_n = V_n \cap (\cup_{i\leq n}\bar{U_i})^{\mathcal{C}}$.

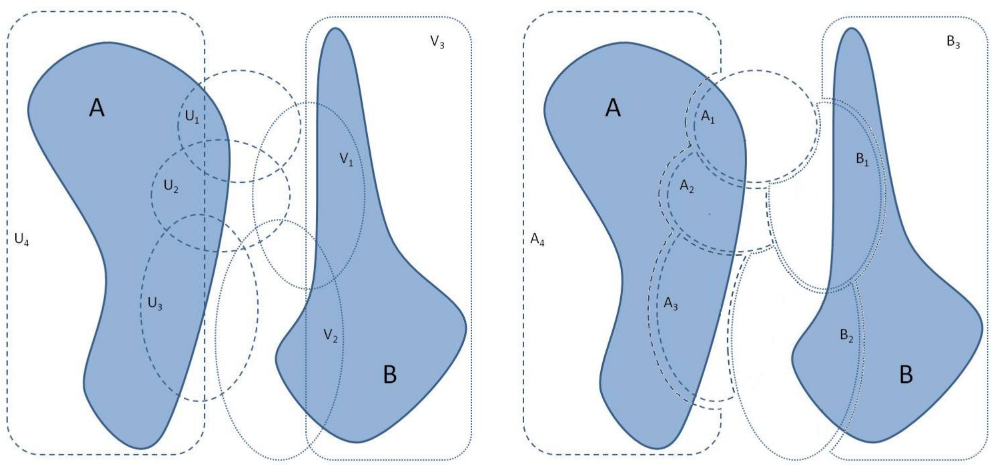

Figure 2: These pictures show how our selection process changes the U_i's to the A_i's and the V_i's to the B_i's. This is not meant as an accurate representation – there are only finitely many sets shown – but rather as an aid to understand how we are picking our new collections of sets.

As a finite union of closed sets, we know that $\cup_{i<n}\bar{V_i}$ will be closed for all $n \in \mathbb{N}$, and therefore its complement will be open. So, as the intersection of two open sets, A_n will be open for all $n \in \mathbb{N}$. The same goes for the B_n's. Therefore, as the union of open sets, $U = \cup_{i\in\mathbb{N}} A_i$ and $V = \cup_{i\in\mathbb{N}} B_i$ will be open sets.

Now, we want to show that $A \subseteq U$ and $B \subseteq V$. Let $a \in A$. Then, since $A \subseteq \cup_{i\in\mathbb{N}} U_i$, $a \in U_j$ for some $j \in \mathbb{N}$. Furthermore, we know that for all $i \in \mathbb{N}$,

$\bar{V}_i \cap A = \emptyset$. So, for all $i \in \mathbb{N}$, $a \notin \bar{V}_i$, and therefore $a \notin \cup_{i<j}\bar{V}_i$. So a is in the complement of that union. Therefore, we see that $a \in U_j \cap (\cup_{i<j}\bar{V}_i)^{\mathcal{C}} = A_j$. So $a \in U$. Since a was arbitrary, we may conclude that $A \subseteq U$. A parallel argument shows that $B \subseteq V$.

It remains to show that $U \cap V = \emptyset$. Let $x \in U$. Then $x \in A_k$ for some $k \in \mathbb{N}$. This means that $x \in U_k \cap (\cup_{i<k}\bar{V}_i)^{\mathcal{C}}$. So $x \notin \bar{V}_i$ for all $i < k$, and therefore $x \notin B_i$ for all $i < k$. Let $m \geq k$. Consider $B_m = V_m \cap (\cup_{i \leq m}\bar{U}_i)^{\mathcal{C}}$. We know that $x \in U_k$, meaning since $k \leq m$, $x \in \cup_{i \leq m}\bar{U}_i$. So x cannot be in the complement of this set, and therefore $x \notin B_m$. We see that for all $i \in \mathbb{N}$, $x \notin B_i$, and therefore $x \notin \cup_{i \in \mathbb{N}} B_i = V$. Thus, we conclude that $U \cap V = \emptyset$. ▮

Theorem 5.7 Every regular, Lindelöf space is normal.

Proof:

Let X be a regular, Lindelöf space. Let A and B be closed subsets of X. Then, by Thm 4.2, for every point $a \in A$, we can find an open set U_a such that $a \in U_a$ and $\bar{U}_a \cap B = \emptyset$. The collection $\{U_a\}_{a \in A}$ will be an open cover for A. Therefore, since X is Lindelöf, by Thm 5.5 we can extract a countable subcover $\{U_i\}_{i \in \mathbb{N}}$. We know then that $A \subseteq \cup_{i \in \mathbb{N}} U_i$ and that for each i in $\mathbb{N}$, $\bar{U}_i \cap B = \emptyset$. We will use the same process to get a countable open cover of B: $\{V_i\}_{i \in \mathbb{N}}$. Now we see that our two open covers satisfy the hypotheses of 5.6. Therefore, we can use the normality lemma to determine that X is normal. ▮

We now have almost all of the pieces necessary to take on the metrization theorem. However, before we get there, we will need one more definition

Definition 5.8 Let X and Y be topological spaces. A function $f : X \to Y$ is a **continuous function** if and only if for every open set $U \in Y$, $f^{-1}(U)$ is open in X.

As a property of functions rather than of sets, continuity will come into play not as a requirement of the space, but ultimately as a way of verifying that the d-metric topology under the metric we create truly is the original topology on the space. To this end, we will need the following characteristic of continuity for the special case where the range of f is a subset of $\mathbb{R}$.

Theorem 5.9 Let $f : X \to \mathbb{R}$ be a continuous function on a topological space X. Then for all $\epsilon > 0$ and $p \in X$, there exists an open set U containing p such that for all $x, y \in U$, $|f(x) - f(y)| < \epsilon$.

Proof:
Let $\epsilon > 0$ and let $p \in X$. Since f is continuous, we know that the inverse image of $(f(p) - \frac{\epsilon}{2}, f(p) + \frac{\epsilon}{2})$ is open, since this is an open interval in $\mathbb{R}$. Furthermore, since the interval contains $f(p)$, $p \in U = f^{-1}[(f(p) - \frac{\epsilon}{2}, f(p) + \frac{\epsilon}{2})]$. Now, let x, $y \in U$. The triangle inequality gives us that

$$|f(x) - f(y)| \leq |f(x) - f(p)| + |f(p) - f(y)|.$$

By the way we've defined our interval,

$$|f(x) - f(p)| + |f(p) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

So $|f(x) - f(y)| < \epsilon$. ∎

6 Urysohn's Metrization Theorem

At this point, we have all of the definitions and supporting theorems necessary to prove Urysohn's characterization of metrizability.

We begin with a lemma:

Theorem 6.1 (**Urysohn's Lemma**) Let X be a normal space. Let A and B be disjoint closed subsets of X. Then there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$.

Before we get into the proof of this lemma, let's take an intuitive look at the situation posed. We have closed, disjoint sets, A and B, and we need to define a function f on the whole space X. We already know f will take the elements of A to 0 and the elements of B to 1; it remains to us to find an equation to send the rest of the elements of X to the interval $[0, 1]$ so that the points somehow 'nearer' A will map to numbers closer to 0, while the points 'nearer' B will map to numbers closer to 1. Normality lets us build a collection of shells around A, open sets one inside the other, to break X into regions closer to or further from A. Of course, we will need infinitely many in order for our f function to be continuous. However, countably infinitely many sets will do – specifically, sets corresponding to the rational numbers between 0 and 1. The sequence of sets will be linked to the sequence of rational numbers by indexing number, with sets corresponding to smaller rational numbers inside sets corresponding to larger rational numbers, as in the figure below.

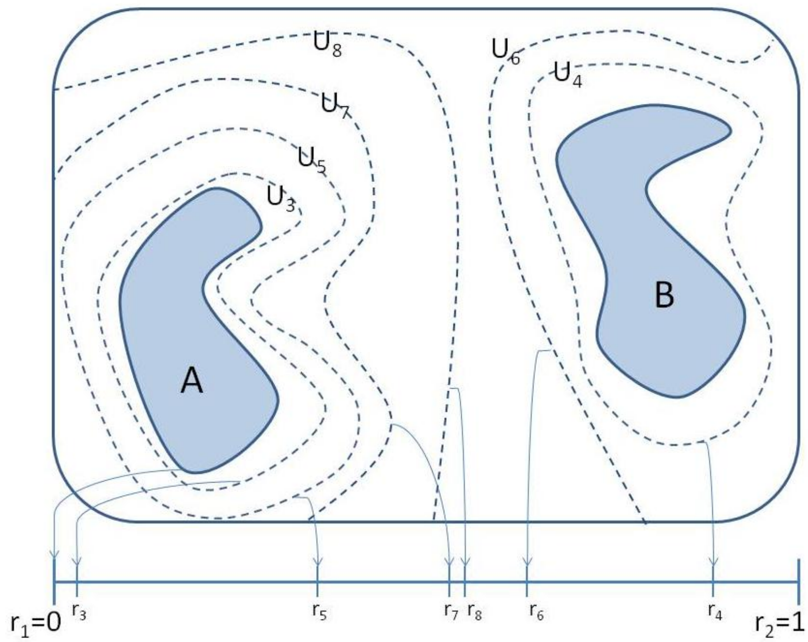

Figure 3: Building a collection of open sets.

Notice that the order of the rational numbers in the sequence does not matter – we will place the corresponding sets in their proper place one by one, using the algorithm described below. Once we have finished constructing our sets, we can use the Least Upper Bound Axiom to define a continuous function.[6]

Proof:

Let A and B be disjoint closed subsets of a normal space X. Let $r_1, r_2, r_3...$ be a distinct sequence of all of the rational numbers in $[0, 1]$, $r_1 = 0$, $r_2 = 1$, and the rest of the rational numbers following in no particular order. Now, we will use this sequence to define a sequence of open sets by induction:

Base Case: Since X is normal, we can find V such that $A \subseteq V$ and $\bar{V} \subseteq B^{\mathcal{C}}$. Let $U_1 = V$ and $U_2 = B^{\mathcal{C}}$. Note that we have $r_1 < r_2$ and $\bar{U_1} \subseteq U_2$.

Induction Hypothesis: Suppose we have open sets U_1, U_2, ..., U_k such that for all i, $j \leq k$, if $r_i < r_j$, then $\bar{U_i} \subseteq U_j$.

Induction Step: Now, we need to define U_{k+1}. Consider r_{k+1}. The set $\{r_p : p \leq k \text{ and } r_p < r_{k+1}\}$ will contain $r_1 = 0$, so as a finite subset of a totally ordered set of real numbers, it will have a greatest element, which we will call r_i. Similarly, choose r_j to be the least element of $\{r_p : p \leq k \text{ and } r_p > r_{k+1}\}$. So we have $r_i < r_{k+1} < r_j$, with no r_p $(p \leq k)$ in the

[6]This proof follows the argument laid out in [M].

interval (r_i, r_j). Now, by the normality of X, we can find an open set U_{k+1} such that $\bar{U_i} \subseteq U_{k+1} \subseteq \bar{U_{k+1}} \subseteq U_j$.
To show that our induction hypothesis holds, let m, $n \leq k+1$ such that $m \neq n$. We can suppose, without loss of generality, that $r_m < r_n$. If neither m nor n equals $k+1$, our induction hypothesis tells us that $\bar{U_m} \subseteq U_n$. If $m = k+1$, then $r_n \geq r_j$. Suppose $r_n = r_j$. Then we already know $\bar{U_m} = \bar{U_{k+1}} \subseteq U_j = U_n$. If $r_n > r_j$, then we use our induction hypothesis to get $U_j \subseteq U_n$. By our induction step, $\bar{U_m} \subseteq U_j$, so we may conclude that $\bar{U_m} \subseteq U_n$. Similarly, if $n = k+1$, then $m \leq r_i$. If $m = r_i$, we get $\bar{U_m} = \bar{U_i} \subseteq U_{k+1} = U_n$. If $r_m < r_i$, we use our induction hypothesis to get $\bar{U_m} \subseteq U_i$. By our induction step, $\bar{U_i} \subseteq U_n$, so we may conclude that $\bar{U_m} \subseteq U_n$. Thus, in any case, our induction hypothesis is preserved.
Now, we want to define our function $f : X \to \mathbb{R}$ as

$$f(x) = \text{lub}(\{0\} \cup \{r_p : x \notin U_p\})$$

for all $x \in X$. Since this set will be a nonempty set of real numbers bounded above by 1 for all $x \in X$, we can be certain that there will always be a unique least upper bound, and thus f will be well defined. It remains to show that f is continuous. First, however, we will need to prove two facts:

Lemma 6.2 If $x \in U_i$, then $f(x) \leq r_i$.
Contrapositive: Suppose $f(x) > r_i$. Since $f(x)$ is the least upper bound for $\{r_p : x \notin U_p\}$, there will exist some r_a such that $x \notin U_a$ and $f(x) > r_a > r_i$. So, by the way we've formed our U sets, $U_i \subseteq \bar{U_i} \subseteq U_a$. So, since $x \notin U_a$, it follows that $x \notin U_i$.

Lemma 6.3 If $x \notin \bar{U_i}$, then $f(x) \geq r_i$.
Suppose $x \notin \bar{U_i}$. Then $x \notin U_i$. So $r_i \in \{r_p : x \notin U_p\}$. Since $f(x)$ is the least upper bound for this set, $f(x) \geq r_i$.

Recall that to show f is continuous, we want to prove that the inverse image of any open set in $[0, 1]$ will be open. To do this, it is sufficient to show that the inverse image of any open interval in $[0, 1]$ is open, since open sets in $\mathbb{R}$ are unions of open intervals. Let (a, b) be an open interval in $[0, 1]$. Let $x \in f^{-1}[(a, b)]$. So $f(x) \in (a, b)$. Therefore, we can find rational numbers r_v and r_w such that $a < r_v < f(x) < r_w < b$. Now, we want to find an open set W such that $x \in W \subseteq f^{-1}[(a, b)]$. We propose $W = U_w \setminus \bar{U_v}$. (Note that, as the intersection of an open set and the complement of a closed set, this will be an open set.)
First, we want to show that $x \in W$. Since $f(x) < r_w$ and $f(x)$ is the least

upper bound of the set $\{r_p : x \notin U_p\}$, we know that r_w cannot be in this set. So $x \in U_w$. Now, since $f(x) > r_v$, we can find a rational number r_b such that $f(x) > r_b > r_v$. So by our construction of the U_p's, $U_v \subseteq \bar{U_v} \subseteq U_b$. The contrapositive of Lemma 6.2 tells us that $f(x) > r_b$ means $x \notin U_b$. So it follows that $x \notin \bar{U_v}$. Thus, we conclude that $x \in U_w \setminus \bar{U_v} = W$.
Now, it remains to show that $W \subseteq f^{-1}[(a,b)]$. So, let $y \in W$. Since $y \in U_w$, Lemma 6.2 gives us that $f(y) \leq r_w$ and since $y \notin \bar{U_v}$, Lemma 6.3 gives us that $f(y) \geq r_v$. So we get $a < r_v < f(y) < r_w < b$. Thus, $y \in f^{-1}[(a,b)]$. So we may conclude $W \subseteq f^{-1}[(a,b)]$.
Thus, f is continuous. ∎

In Urysohn's Metrization Theorem, we will use functions like the one we've just constructed to define a metric on X.[7]

Theorem 6.4 (**Urysohn's Metrization Theorem**) Every second countable, regular, T_1 space is metrizable.

Proof:

Let X be a second countable, regular, T_1 space. Second countability tells us that X is Lindelöf (Thm 5.4), so by Thm 5.7, as a regular, Lindelöf space, X is normal.
Now, since X is second countable, it will have a countable basis, $\mathcal{B} = \{B_1, B_2, ...\}$. Consider all pairs of elements of the basis, B_i, B_j such that $\bar{B_i} \subseteq B_j$. This collection will be countable, so we can write the pairs as P_1, P_2,.... Urysohn's Lemma tells us that for each pair, $P_n = (\mathrm{B}_{n_i}, \mathrm{B}_{n_j})$ we can define a continuous function f_n such that $f_n(\bar{B_{ni}}) = 0$ and $f_n(B_{nj}^{\mathcal{C}}) = 1$. Now, define a function $d : X \times X \to \mathbb{R}$, such that for any x, $y \in X$,

$$d(x,y) = \sum_{1}^{\infty} 2^{-n} |f_n(x) - f_n(y)|.$$

First, we want to show that d is a metric:

1. We see from the definition of d that d maps pairs of points in X to the nonnegative real numbers.

2. $\underline{d(x,y) = 0 \Leftrightarrow x = y}$.

    - ($\Rightarrow$) Suppose $x \neq y$. Since X is T_1, $\{x\}$ and $\{y\}$ are closed sets. So, by normality, since we have a point x that is not in a closed set

[7] This proof follows the argument laid out by Whyburn and Duda; see [W-D].

$\{y\}$, by Lemma 4.6, we can find basis elements B^* and B° such that $x \in B^* \subseteq \bar{B^*} \subseteq B^\circ \subseteq \{y\}^{\mathcal{C}}$. Therefore, one of our defined P_i's corresponds to this pair of basis elements, which we will call P_k. So our corresponding function, f_k, will have $f_k(u) = 0$ for all $u \in B^*$ and $f_k(v) = 1$ for all $v \in \{B^\circ\}^{\mathcal{C}}$. So, since $x \in B^*$ and $y \in (B^\circ)^{\mathcal{C}}$, $|f_k(x) - f_k(y)| = |0 - 1| = 1$. This means that

$$d(x,y) = \sum_1^\infty 2^{-n}|f_n(x) - f_n(y)| \geq 2^{-k} > 0.$$

- ($\Leftarrow$) Suppose $x = y$. Then by the well-defined nature of f, $f_n(x) = f_n(y)$ for all $n \in \mathbb{N}$. So

$$d(x,y) = \sum_1^\infty 2^{-n}|f_n(x) - f_n(y)| = \sum_1^\infty 2^{-n}(0) = 0.$$

3. $\underline{d(x,y) = d(y,x) \text{ for all } x,\ y \in X}$. Let x, $y \in X$. Then

$$d(x,y) = \sum_1^\infty 2^{-n}|f_n(x) - f_n(y)| = \sum_1^\infty 2^{-n}|f_n(y) - f_n(x)| = d(y,x).$$

4. $\underline{\text{For } x,\ y,\ z \in X,\ d(x,y) + d(y,z) \geq d(x,z)}$. Let x, y, $z \in X$.

$$\begin{aligned} d(x,y) + d(y,z) &= \sum_1^\infty 2^{-n}|f_n(x) - f_n(y)| + \sum_1^\infty 2^{-n}|f_n(y) - f_n(z)| \\ &= \sum_1^\infty 2^{-n}(|f_n(x) - f_n(y)| + |f_n(y) - f_n(z)|) \\ &\geq \sum_1^\infty 2^{-n}|f_n(x) - f_n(z)| \\ &= d(x,z). \end{aligned}$$

At this point, we will pause in the proof to consider the metric we've just defined. Initially, it seems strange that the order of the f_n's is arbitrary, so that the 2^{-n} gives more or less weight to a given f_n depending on its indexing number. However, what matters is that the f_n's are always in the same order. It is important to realize that distance is not something inherent in a set of points. There are many ways to define a metric – in the Euclidean Plane, for example, there are the usual Euclidean metric, the taxicab metric, and trivial metric, just as a starting point. Therefore, trying to visual distance geometrically is counterproductive. Instead, distance is defined on two points based on the ways they can be separated by open sets, which in turn depends on the topology. We will now return to the proof.

Now that we've established that d is a metric for X, it remains to show that the topology created by our basis $\mathcal{B}$ is the d-metric topology. To do this, we will use the alternate condition for metrizability discussed in Theorem 3.13 and the alternate condition for continuity discussed in Theorem 5.9 So, we want to show that, under d, the following statement holds:

p is a limit point of M in $(X, \mathcal{T}) \Leftrightarrow$ for all $\epsilon > 0$, there exists $m \in M$ such that $0 < d(p, m) < \epsilon$.

($\Rightarrow$) Suppose p is a limit point of M in $(X, \mathcal{T})$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $2^{-N} < \frac{\epsilon}{2}$. By the continuity of the f_n's, for each $n \leq N$, we can find an open U_n containing p such that for all x, $y \in U_n$, $|f_n(x) - f_n(y)| < \frac{\epsilon}{4N}$. Let U be the intersection of the U_n's for $n \leq N$. (Since this is a finite intersection of open sets, U will be open.) Since U is an open set containing p, the definition of a limit point tells us we can find $m \in M \cap (U \setminus \{p\})$. So $0 < d(p, m)$. It remains to show that $d(p, m) < \epsilon$.

$$\begin{aligned}
d(p,m) &= \sum_{i=1}^{\infty} 2^{-n}|f_i(p) - f_i(m)| \\
&= \sum_{i=1}^{N} 2^{-i}|f_i(p) - f_i(m)| + \sum_{i=N+1}^{\infty} 2^{-i}|f_i(p) - f_i(m)| \\
&< \sum_{i=1}^{N} 2\frac{\epsilon}{4N} + \sum_{i=N+1}^{\infty} 2^{-i} \\
&< \frac{\epsilon}{2} + 2^{-N}\sum_{i=1}^{\infty} 2^{-i} \\
&= \frac{\epsilon}{2} + 2^{-N} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

($\Leftarrow$) Suppose that for all $\epsilon > 0$, there exists $m \in M$ such that $0 < d(p, m) < \epsilon$. Suppose p is not a limit point for M. So there exists an open set U containing p such that $(U \setminus \{p\}) \cap M = \emptyset$. So, since $p \in U^{\mathcal{C}}$, by Lemma 4.6, we can find basis elements B^* and B° such that $p \in B^* \subseteq \bar{B}^* \subseteq B^\circ \subseteq (U^{\mathcal{C}})^{\mathcal{C}} = U$. This means that this B^*, B° pair is P_n for some $n \in \mathbb{N}$. So we have a function f_n

such that $|f_n(p) - f_n(q)| = 1$ for all $q \in (B^\circ)^{\mathcal{C}}$. Let $m \in M$. If $m = p$, then $d(p,m) = 0$. If $m \neq p$, then $m \notin U$. Therefore $m \notin B^\circ$, so $m \in (B^\circ)^{\mathcal{C}}$. Thus, $d(p,m) \geq 2^{-n}$. So for all $m \in (M \backslash \{p\})$, $d(p,m)$ is not in the interval $(0, 2^{-n})$. But this is a contradiction, since we can choose $0 < \epsilon < 2^{-n}$. Therefore, p must be a limit point for M. ∎

7 Conclusion

Urysohn's Metrization Theorem is an highly useful result for determining whether a given topological space X is metrizable. Rather than trying to construct a metric for X – which can be quite difficult – we need only verify that X is second countable, regular, and T_1. Another benefit of this theorem is that it is so accessible. Certainly it requires a solid foundation in analysis, but in comparison to other metrization theorems, the mathematics involved is relatively simple.

However, one thing that might bother us about Urysohn's characterization of metrizable spaces is that it is not an if and only if statement. In fact, recall that we listed any set X under the discrete topology as a metric space. Yet we also pointed out that if X is uncountable, X under the discrete topology is not a second countable space. So already we have seen an example of a metrizable space that is not second countable.

One famous necessary and sufficient characterization of metrizability is the **Nagata-Smirnov Theorem**:

> A topological space is metrizable if and only if it is T_1 and regular, and has a σ-locally finite basis.

So, we do not have to have second countability to get metrizability; instead, the theorem gives the requirement of a σ-locally finite basis as a weaker alternative.

Definition 7.1 Let $\mathcal{T}$ be a topology on a set X with a basis $\mathcal{B}$. $\mathcal{B}$ is a **locally finite basis** if for each $x \in X$, there exists some open set containing x that intersects only finitely many elements of $\mathcal{B}$. More generally, $\mathcal{B}$ is **σ-locally finite** if it is a countable union of locally finite subsets of X.

So, X need not have a countable basis; it need only have a basis that is a countable union of locally finite sets. In a crude sense, we can think of metrizability as depending on a space X having "enough" open sets, but "not too many." The idea of the "amount" of open sets is intended as an intuitive tool for understanding, rather than a rigorous definition, and clearly it will vary from set to set. In general, though, the normality requirement of a metric space guarantees that there are "enough" open sets – this is what allows us to continue to take nested sets in Urysohn's Lemma. If we didn't have normality, we might run out of sets, in which case we would not have gotten continuity for our function f. The "not too many" condition appears as the second countable requirement of Urysohn's Metrization Theorem, and as the σ-locally finite basis requirement of the Nagata-Smirnov Theorem. The intuition behind this condition, however, and the proof of the Nagata-Smirnov Theorem, are beyond the scope of this paper. For more information, consult James R. Munkres' *Topology: A First Course*.

8 Acknowledgements

I would like to thank my advisors, Prof. Carol Schumacher and Prof. Chris Camfield, for all of their help – reading my drafts and answering my (many many) questions. Prof. Camfield could come up with an example of any type of space I threw at him (provided that such a space existed), and Prof. Schumacher patiently talked me through the tougher results of the paper, even on the days that I trudged into her office, flung myself down in her chair, and declared I didn't even know what topology *was*. It was lovely working with them, and I doubt that this exercise could have existed without them.

9 References

[**B**] Bing, R. H. "Elementary Point Set Topology." *The American Mathematical Monthly* Vol. 67 No. 7 (1960): 1-58. Print.

[**M**] Munkres, James R. *Topology: A First Course*. Englewood Cliffs: Prentice-Hall, 1975.

[**Sch**] Schumacher, Carol S. *Closer and Closer: Introducing Real Analysis*. Sudbury, MA: Jones and Bartlett, 2008.

[**St**] Starbird, Michael. Moore Method Topology Notes (unpublished).

[**S-S**] Steen, Lynn Arthur, and J. Arthur Seebach, Jr. *Counterexamples in Topology.* 2nd Ed. New York: Springer-Verlag, 1970.

[**W-D**] Whyburn, Gordon, and Edwin Duda. *Dynamic Topology.* New York: Springer-Verlag, 1979.