Part I: Techniques of Integration, Sequences and Series (100 points)

1. The answers are:
   (a) (10 points) (Integration by Parts) \( \int 2xe^x \, dx = 2xe^x - 2e^x + C \)
   (b) (10 points) (Partial Fractions) \( \int \frac{dx}{x^2 - 1} = 1/2 \int \frac{dx}{x-1} - 1/2 \int \frac{dx}{x+1} \)

2. (10 points) First separate the variables \( x \) and \( y \): \( 3y^2 \, dy = dx \), then integrate: \( y^3 = x + C \).
   Since \( y(0) = 2 \), it follows that \( 2^3 = 0 + C \) (\( C = 8 \)) and therefore \( y = \sqrt{x+8} \).

3. (10 points) \( \int_0^\infty \frac{dx}{1+x^2} = \lim_{b \to \infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b \to \infty} \tan^{-1} b = \frac{\pi}{2} \)

4. (5 points) \( f'(x) = d/dx \int_1^x \frac{dt}{t} = 1/x \) (Fundamental Theorem of Calculus)

5. (5 points) \( \int_0^4 f(x) \, dx \approx 1/2(0 + 0 + 2 \cdot 2 + 2 \cdot 6 + 12) = 14 \) (Trapezoidal rule)
   \( \int_0^4 f(x) \, dx \approx 1/3(0 + 0 + 2 \cdot 2 + 4 \cdot 6 + 12) = 40/3 \) (Simpson’s rule)

6. (10 points) \( \sum_{n=0}^\infty \frac{4 \cdot (-3)^n}{5^n} = \frac{4}{1 - (-3/5)} = 5/2 \)

7. The answers are:
   (a) (8 points) Convergent by comparison to \( \sum_{k=1}^\infty \frac{\sqrt{k}}{k^2} = \sum_{k=1}^\infty \frac{1}{k^{3/2}} \), which is a convergent \( p \)-series.
   (b) (8 points) Conditionally convergent. Converges by alternating series test, but \( \sum_{k=1}^\infty \frac{1}{\sqrt{k}} = \sum_{k=1}^\infty \frac{1}{\sqrt{k}} \), a divergent \( p \)-series.
   (c) (8 points) Convergent by Ratio Test.

8. (8 points) Using Ratio Test, converges absolutely for \( \lim_{n \to \infty} \frac{2(n+1) + 1 \cdot |x-5|^{n+1}}{3^{n+1}} \cdot \frac{3^n}{2n+1 \cdot |x-5|^n} = \frac{|x-5|}{3} < 1 \), which is \( |x-5| < 3 \), or \( 2 < x < 8 \).
Now we check endpoint convergence: At \( x = 2, \sum_{n=0}^{\infty} \frac{2n+1}{3^n}(x-5)^n = \sum_{n=0}^{\infty} (-1)^n(2n+1), \)
which diverges by the \( n^{th} \)-term Test for Divergence. At \( x = 5, \sum_{n=0}^{\infty} \frac{2n+1}{3^n}(x-5)^n = \sum_{n=0}^{\infty} 2n+1 \) which also diverges by the \( n^{th} \)-term Test for Divergence.
Hence, the interval of convergence is \( (2, 8) \) or \( 2 < x < 8 \).

9. (8 points) \( \sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \ldots \)
\( \sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \ldots \)
\( P_6(x) = x^2 - \frac{x^6}{6} \) (Taylor polynomial of degree 6)

**Part II: Applications of Integration** (50 points)

10. (10 points) The region bounded by \( y = x^2 - 1 \) and \( y = x + 1 \):

\[
\begin{align*}
\text{Area} &= \int_{-1}^{2} (x + 1) - (x^2 - 1) \, dx = 27/6
\end{align*}
\]

11. Here is the area being rotated:
(a) (10 points) (Washer with outer radius \( R = x \) and inner radius \( r = x^2 \))

Volume = \( \int_{0}^{1} \pi (x^2 - x^4) \, dx \)

(b) (10 points) (Washer with outer radius \( R = 2 - x^2 \) and inner radius \( r = 2 - x \))

Volume = \( \int_{0}^{1} \pi ((2 - x^2)^2 - (2 - x)^2) \, dx \)

12. (10 points) Solve the differential equation: \( \frac{dN}{dt} = k(10,000 - N) \), \( N_0 = 1000 \) & \( N_1 = 2000 \)

\(|10,000 - N| = Ce^{-kt}\)

\( N_0 = 1000 \) implies \( C = 9,000 \) and \( N_1 = 2000 \) implies \( k = \ln(3\sqrt{2}/4) \) thus five month later \( N = 10,000 - 9,000(2/3\sqrt{2})^5 \approx 3296 \) people will have had the disease.

13a (10 pts.) Work on bucket = 80,000 foot pounds, work on rope = \( \int_{0}^{800} 2(800 - h) \, dh = 640,000 \) foot pounds; altogether 720,000 foot pounds of work is done to bring the load of ore to the surface.

13b (10 pts.)

(a) Fraction of the population between 50 and 100 years old = area under the density graph for \( 50 \leq x \leq 100 = \int_{50}^{100} \rho(x) \, dx = 3/8 = 37.5\% \)

(b) If \( M = \) median age of the United States population then \( \int_{0}^{M} \rho(x) \, dx = 1/2 \) thus \( M = 40. \)