Contents

1 Introduction

We often think of fractals as self-similar objects which are the result of an iterative process, self-similar meaning that if we apply the actions of the process we will have an object that looks exactly like the original. Not only can we apply these actions to the object, but also to individual points that comprise the object. Let's look at a very simple example. Consider the process defined by the function $f(x) = \frac{1}{3}x$ on the real numbers. This process takes an element of the real numbers and divides it by three. Let's apply our process to 1. $f(1) = \frac{1}{3}$. Now suppose we apply our process to $\frac{1}{3}$. $f(\frac{1}{3}) = \frac{1}{9}$. Notice that this is the result of applying our process twice to 1, or, in other words, we iterate f on 1 twice, and the iterates of 1 are $\frac{1}{3}, \frac{1}{9}$. Suppose we iterate f on 1 infinitely many times, then the iterates of f form a sequence $(\frac{1}{3^n})$. This sequence converges to zero and note that if we apply our process to zero, then we get zero back. We call the sequence $(\frac{1}{3^n})$ the orbit of 1 under f. Now we can certainly look at this orbit and it is obvious that it converges to zero, but suppose we were to compute the orbit on a calculator. The calculator can only keep a fixed number of decimal places, so there is error introduced in the calculations. Is this a problem? Should we be worried? In fact, it is not as bad as we might think. The Shadowing Theorem rescues us. It states that for any errorful orbit there will exist an exact orbit of another point within a certain distance from the errorful orbit, or, in other words, an orbit that "shadows" the errorful one. The distance depends upon the error that we introduce. To understand the Shadowing Theorem, we must first understand a metric space called the Hausdorff metric space.

2 Hausdorff Metric Space

The Hausdorff metric space is vital to theorems proved later in this paper, so it is important that we understand what it is and know some basic properties of the space. The Hausdorff metric space is denoted by $(\mathcal{H}(\mathbf{X}), h)$. We define the space $\mathcal{H}(\mathbf{X})$ as follows.

Definition 2.1 Let $(\mathbf{X}, d)$ be a complete metric space[1]. Then $\mathcal{H}(\mathbf{X})$ denotes the space whose points are the compact[2] subsets of $\mathbf{X}$, other than the empty set.[3]

Before we formally define the distance metric h we must make some preliminary definitions. First we define the distance between points in $\mathbf{X}$ and points in $\mathcal{H}(\mathbf{X})$.

Definition 2.2 Let $(\mathbf{X}, d)$ be a complete metric space, $x \in \mathbf{X}$, and $B \in \mathcal{H}(\mathbf{X})$. Define

$$d(x, B) = \min\{d(x, y) : y \in B\}.$$

We will call $d(x, B)$ the distance from the point x to the set B.[4]

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[1] See Definition 7.4 in the Appendix

[2] See Definition 7.7 in the Appendix

[3] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 27.

[4] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 27.

Now we use this definition to define the distance between two points in $\mathcal{H}(\mathbf{X})$.

Definition 2.3 Let $(\mathbf{X}, d)$ be a complete metric space. Let $A, B \in \mathcal{H}(\mathbf{X})$. Define

$$d(A, B) = \max\{d(x, B) : x \in A\}.$$

We will call $d(A, B)$ the distance from the set $A \in \mathcal{H}(\mathbf{X})$ to the set $B \in \mathcal{H}(\mathbf{X})$.[5]

Notice that this definition is not symmetric, meaning that the distance from A to B is not necessarily the same as the distance from B to A (See Example 2.5).

Now we are ready to define the Hausdorff distance metric between points A and B in $\mathcal{H}(\mathbf{X})$.

Definition 2.4 Let $(\mathbf{X}, d)$ be a complete metric space. Then

$$h(A, B) = \max\{d(A, B), d(B, A)\}.$$

We will call $h(A, B)$ the *Hausdorff distance* between points A and B in $\mathcal{H}(\mathbf{X})$.[6]

It can be show that $(\mathcal{H}(\mathbf{X}), h)$ is, indeed, a metric space[7]. To gain some intuition about the Hausdorff metric let's look at an example where $d(A, B) \neq d(B, A)$.

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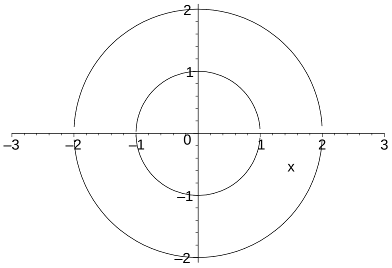
[5] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 29.

[6] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 32.

[7] See Theorem 7.2

Example 2.5 Let $\mathbf{X}$ be $\mathbb{R}^2$, $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$, and $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. These are two disks centered at the origin, one of radius two, denote this one as A, and one of radius one, denote this one as B.



The distance from A to B is the maximum of the distances from point in A to B. Note that the farthest points of disk A from disk B are those on the boundary. If we let $x = (2,0)$, which is on the boundary of A, then $d(x,B) = 1$ because the closest point in the disk B to $(2,0)$ is $(1,0)$. So $d(A,B) = 1$. Now look at the distance from B to A. Notice that all the points of B are also in A. Consider the point $(0,0)$ as an element of B, then $d(x,A) = 0$ because the closest point of A to $(0,0)$ in B is $(0,0)$ in A. So $d(B,A) = 0$ and $d(A,B) \neq d(B,A)$. The Hausdorff distance between the disks A and B is given by the following.

$$h(A,B) = \max\{d(A,B), d(B,A)\} = \max\{1,0\} = 1$$

This next theorem addresses the question: What does it mean for two elements of $\mathcal{H}(\mathbf{X})$ to be close together?

Theorem 2.6 Let A and B belong to $\mathcal{H}(\mathbf{X})$ where $(\mathbf{X}, d)$ is a metric space. Let $\epsilon > 0$. Then $h(A,B) \leq \epsilon$ if and only if $A \subset B + \epsilon$ and $B \subset A + \epsilon$.

Proof:

Begin by showing $d(A, B) \leq \epsilon$ if and only if $A \subset B + \epsilon$.

- Suppose $d(A, B) \leq \epsilon$. Then $\max\{d(a, B) : a \in A\} \leq \epsilon$ implies $d(a, B) \leq \epsilon$ for all $a \in A$. Therefore, for every $a \in A$ we have $a \in B + \epsilon$. Hence $A \subset B + \epsilon$.

- Suppose $A \subset B + \epsilon$. Consider $d(a, B) = \max\{d(a, B) : a \in A\}$. Let $a \in A$. Since $A \subset B + \epsilon$, there is $b \in B$ such that $d(a, b) \leq \epsilon$, so $d(a, B) \leq \epsilon$. This is true for all $a \in A$, therefore, $d(A, B) \leq \epsilon$.

By a parallel argument we can show that $d(B, A) \leq \epsilon$ if and only if $B \subset A + \epsilon$.

Therefore, if $d(A, B) \leq \epsilon$ and $d(B, A) \leq \epsilon$, then $A \subset B + \epsilon$ and $B \subset A + \epsilon$, but $d(A, B) \leq \epsilon$ and $d(B, A) \leq \epsilon$ implies that $h(A, B) \leq \epsilon$. So $h(A, B) \leq \epsilon$ implies $A \subset B + \epsilon$ and $B \subset A + \epsilon$. If $A \subset B + \epsilon$ and $B \subset A + \epsilon$, then $d(A, B) \leq \epsilon$ and $d(B, A) \leq \epsilon$, but this means that $h(A, B) \leq \epsilon$.

Thus $h(A, B) \leq \epsilon$ if and only if $A \subset B + \epsilon$ and $B \subset A + \epsilon$.[8] ▮

We can show that given any complete metric space $(\mathbf{X}, d)$, the metric space $(\mathcal{H}(\mathbf{X}), h)$ is complete.

Theorem 2.7 Let $(\mathbf{X}, d)$ be a complete metric space. Then $(\mathcal{H}(\mathbf{X}), h)$ is a complete metric space. Moreover, if $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}(\mathbf{X})$, then

$$A = \lim_{n \to \infty} A_n$$

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[8] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 34.

can be characterized as follows:

$$A = \Big\{x \in \mathbf{X} : x = \lim_{n\to\infty} x_n \text{ where } \{x_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence such that } x_n \in A_n\Big\}$$

Proof:

This is theorem and its proof appears in the Appendix as Theorem 7.11 ▮

Now that we have established this information about the Hausdorff metric space we can move on to iterated function systems, which are the processes that we will be using.

3 Iterated Function Systems

An iterated function system is simply defined as a set of special functions called contractions, so let's begin by gaining an understanding of contractions.

3.1 Contractions

A contraction is defined as follows.

Definition 3.1 Let $(\mathbf{X}, d)$ be a metric space. A function $f : \mathbf{X} \to \mathbf{X}$ with the property that for some real number $k < 1$,

$$d(f(x), f(y)) \leq kd(x, y)\text{for all } x, y \in \mathbf{X}$$

is called a *contraction* of $(\mathbf{X}, d)$, and k is called the *contraction factor.* [9]

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[9] Bryant, Victor. Metric spaces: iteration and application. 1985. p. 57.

We have already seen an example of a contraction in the introduction. Let's show that $f(x) = \frac{1}{3}x$ is a contraction of $(\mathbb{R}\ , d)$ where $\mathbb{R}$ is the set of real numbers and d is the Euclidean metric.

Example 3.2 . To show that $f(x) = \frac{1}{3}x$ is a contraction we must show that there exists $k < 1$ such that

$$d\left(\frac{1}{3}x, \frac{1}{3}y\right) \leq kd(x, y)\text{for all } x, y \in \mathbb{R}$$

Consider $k = \frac{1}{3}$. Now we have to show that $d\left(\frac{1}{3}x, \frac{1}{3}y\right) \leq \frac{1}{3}d(x, y)$ for all $x, y \in \mathbb{R}$.

$$\begin{aligned} d\left(\frac{1}{3}x, \frac{1}{3}y\right) &= \left|\frac{1}{3}x - \frac{1}{3}y\right| \\ &= \left|\frac{1}{3}(x - y)\right| \\ &= \frac{1}{3}|x - y| \\ &= \frac{1}{3}d(x, y) \end{aligned}$$

Therefore, $d(f(x), f(y)) \leq \frac{1}{3}d(x, y)$, so f is a contraction of $(\mathbb{R}\ , d)$.

What about functions that are not contractions? Consider $f(x) = x^2$.

Example 3.3 The mapping $f(x) = x^2$ on the metric space $(\mathbb{R}\ , d)$ is not a contraction, and to show this it is sufficient to show that there exist $x, y \in \mathbb{R}$ such that there is no $k < 1$ such that $d(f(x), f(y)) \leq kd(x, y)$. Consider $2, 4 \in \mathbb{R}$. $d(2, 4) = 2$ and $d(f(x), f(y)) = d(4, 16) = 12$. The only way for 12 to be less than or equal to $k4$ is if $k \geq 3$, but we know that k must be less than one. Therefore, $f(x) = x^2$ is not a contraction.

We need not limit ourselves to applying the contraction just once. In fact, we can apply, or iterate, a contraction, f, as many times as we want.

Definition 3.4 The *forward iterates* of f are functions $f^n : \mathbf{X} \to \mathbf{X}$ defined by $f^0(x) = x$, $f^1(x) = f(x)$, $f^{n+1} = f \circ f^n(x) = f(f^n(x))$ for $n \in \mathbb{N}$. If f is invertible, then the *backward iterates* of f are transformations $f^{-m} : \mathbf{X} \to \mathbf{X}$ defined by $f^{-m}(x) = (f^m)^{-1}(x)$.[10]

Let's look at the contraction from Example 3.2 and find some of its iterates.

Example 3.5 The forward iterate, f^3, of $f(x) = \frac{1}{3}x$ is

$$\begin{aligned} f^{(3)}(x) &= f \circ f^{(2)}(x) \\ &= f \circ (f \circ f)(x) \\ &= \frac{1}{3}(\frac{1}{3}(\frac{1}{3}(x))) \\ &= \frac{1}{27}(x) \end{aligned}$$

The backward iterate, f^{-3}, of $f(x) = \frac{1}{3}x$.

$$\begin{aligned} f^{(-3)} &= (f^3)^{-1}(x) \\ &= (\frac{1}{27}x)^{-1} \\ &= 27x. \end{aligned}$$

Notice that for the function $f(x) = \frac{1}{3}$ $f(0) = 0$. The point 0 has a special name. It's called the fixed point and is formally defined as follows.

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[10] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 43.

Definition 3.6 Let f be a function. The point x is a *fixed point* of f if $f(x) = x$.[11]

There are also points that repeat on cycles. These points are called periodic points and are formally defined as follows.

Definition 3.7 Let f be a function. The point x is a periodic point of f with period k if $f^k(x) = x$.[12]

There are points that after a fixed number of iterations of the function result in a periodic or fixed point. These points are called eventually fixed and eventually periodic and are formally defined as follows.

Definition 3.8 Let f be a function. The point x is an eventually fixed point of f if there exists N such that $f^{n+1}(x) = f^n(x)$ whenever $n \geq N$. The point x is eventually periodic with period k if there exists N such that $f^{n+k}(x) = f^n(x)$ whenever $n \geq N$.[13]

Now that we have an understanding of contractions we are ready to formally define iterated function systems and look at some examples.

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[11] Holmgren, Richard A. *A first course in discrete dynamical system*, second edition. 1996. p. 31.

[12] Holmgren, Richard A. *A first course in discrete dynamical system*, second edition. 1996. p. 33.

[13] Holmgren, Richard A. *A first course in discrete dynamical system*, second edition. 1996. p. 34.

3.2 Iterated Function Systems

The formal definition for an iterated function system is:

Definition 3.9 An *iterated function system* (often abbreviated IFS) consists of a complete metric space $(\mathbf{X}, d)$ together with a finite set of contractions denoted by $\omega_n : \mathbf{X} \to \mathbf{X}$, with respective contraction factors k_n, for $n = 1, 2, 3, ..., N$. The notation for the iterated function system just described is $\{\mathbf{X}; \omega_n, n = 1, 2, 3, ...N\}$ and its contraction factor is $k = \max\{k_n : n = 1, 2, 3, ..., N\}$.[14]

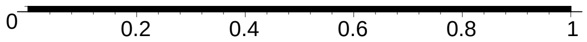
Let's look at an example of an IFS

Example 3.10 Let's examine the forward iterates of the IFS

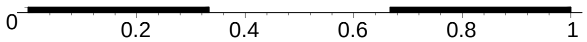
$$\{[0, 1]; \frac{1}{3}x, \frac{1}{3}x + \frac{2}{3}\}$$

Solution:

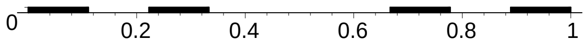
Initially the interval looks like this.



The first iteration of the IFS on the interval $[0, 1]$ looks like this



The second iteration of the IFS on $[0, 1]$ looks like this.



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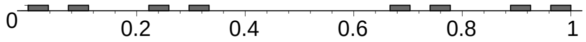
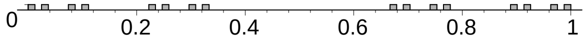
[14] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 80.

The third iterated of the IFS on $[0, 1]$ looks like this.



The fourth iterated of the IFS on $[0, 1]$ looks like this.



The Cantor Set, or Cantor Dust, is a fractal figure that is a subset of the metric space $[0, 1]$. It is obtained by successive deletion of the middle third open sub-interval of each existing interval beginning with $[0, 1]$. If we examine the figures above we see that the IFS performs exactly this action. So the limit of the forward iterates of the IFS should be the Cantor Set.

To formalize the idea that the limit of the iterates of the above IFS is the Cantor Set, we need the idea of attractors. Notice that the result of applying the IFS to the Cantor Set is the Cantor Set. So the Cantor Set is also a fixed point for the IFS. These two ideas give us some intuition about how an attractor should be defined. It should be the limit of the iterates of the IFS and it should be a fixed point of the IFS. To formally define the attractor we are first going to establish the following theorem called the Contraction Mapping Principle.

Theorem 3.11 Let $f : \mathbf{X} \to \mathbf{X}$ be a contraction of the complete metric space $(\mathbf{X}, d)$ with contraction factor k. Then f has a unique fixed point. Furthermore, if x_1 is any point of $\mathbf{X}$, then the sequence $x_1, x_2 = f(x_1), x_3 = f(x_2)$... converges to that fixed point.

Proof:

Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$ $k^{n-1} < \frac{\epsilon(1-k)}{d(x_1,x_2}$. Let x_1 be any point of $\mathbf{X}$ and, as usual, let $x_2 = f(x_1), x_3 = f(x_2)$... Since $(\mathbf{X}, d)$ is complete, to show that (x_n) converges it is sufficient to show that (x_n) is Cauchy. Begin by considering the distance between two consecutive terms x_n and x_{n+1}.

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \\
&\leq kd(d(x_{n-1}, x_n)) = kd(f(x_{n-2}), f(x_{n-1})) \\
&\leq k^2 d(d(x_{n-2}, x_{n-1})) = k^2 d(f(x_{n-3}), f(x_{n-2})) \\
&\quad \vdots \\
&\leq k^{n-1} d(x_1, x_2)
\end{aligned}$$

Now we can use information about consecutive terms to examine the more general case of $d(x_n, x_n)$ where $N < n < m$.

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m) \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m) \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_m) \\
&\quad \vdots \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-2}, x_{m-1}) + d(x_{m-1}, x_m)
\end{aligned}$$

We already established an upper bound for consecutive terms of the sequence, so we can use that information to establish the following.

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m)$$

$$\begin{aligned}
&\leq k^{n-1}d(x_1, x_2) + k^n d(x_1, x_2) + ... + k^{m-2}d(x_1, x_2) \\
&= k^{n-1}(d(x_1, x_2) + kd(x_1, x_2) + ... + k^{m-n-1}d(x_1, x_2) \\
&= k^{n-1}(1 + k + k^2 + ... + k^{m-n-1})(d(x_1, x_2)) \\
&\leq k^{n-1}(1 + k + k^2 + ...)(d(x_1, x_2)) \\
&= \frac{k^{n-1}}{1-k}d(x_1, x_2) \\
&< \frac{\frac{\epsilon(1-k)}{d(x_1,x_2)}}{1-k}d(x_1, x_2) \\
&= \epsilon.
\end{aligned}$$

Thus (x_n) converges to a limit we will denote as x. Since f is a contraction and the metric space is complete, we also know that the sequence $(f(x_n))$ is Cauchy and will converge to $f(x)$. But we defined $f^n(x_1) = f(x_n)$, so (x_n) converges to $f(x)$ as well. Since limits of sequences are unique, $f(x) = x$. Thus the limit, x is indeed a fixed point of f.

Now we must show that the fixed point of f is unique. Suppose there exist $x' \neq x$ such that $f(x) = x$ and $f(x') = x'$. Notice that $d(f(x), f(x')) = d(x, x')$. This implies that f is not a contraction since the contraction factor would have to be 1. Thus we have a contradiction. Hence the fixed point of f is unique.[15] ∎

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[15] Bryant, Victor. Metric spaces: iteration and application. 1985. p. 58-60.

As a corollary to this theorem we have the following result where the metric space is $(\mathcal{H}(\mathbf{X}), h)$ and the contraction mapping is an IFS.

Corollary 3.12 Let $\{\mathbf{X}; \omega_n, n = 1, 2, ...N\}$ be an IFS with contraction factor k. The transformation $W : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$ defined by

$$W(B) = \bigcup_{n=1}^{N} \omega_n(B)$$

for all $B \in \mathcal{H}(\mathbf{X})$, is a contraction mapping on the complete metric space $(\mathcal{H}(\mathbf{X}), h(d))$ with contraction factor k. That is

$$h(W(B), W(C)) \leq k \cdot h(B, C)$$

for all $B, C \in \mathcal{H}(\mathbf{X})$. Its unique fixed point, $A \in \mathcal{H}(\mathbf{X})$, obeys

$$A = W(A) = \bigcup_{n=1}^{N} \omega_n(A)$$

and is given by $A = \lim_{n\to\infty} W^n(B)$ for any $B \in \mathcal{H}(\mathbf{X})$.[16]

Now we can define the attractor as follows.

Definition 3.13 The fixed point $A \in \mathcal{H}(\mathbf{X})$ described in Theorem 3.12 is called the *attractor* of the IFS. [17]

Since $A \in \mathcal{H}(\mathbf{X})$ it is compact and $\mathcal{H}(\mathbf{X})$ is complete, by Theorem 7.8A is totally bounded. Therefore, we can define the diameter of the attractor as follows and note that it will always be finite.

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[16] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 81.

[17] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 81.

Definition 3.14 The *diameter* of $A \in \mathcal{H}(\mathbf{X})$ is

$$\sup\{d(x,y) : x, y \in A\}.$$

The diameter of A is denoted $\text{diam}(A)$.[18]

Let's look again at Example 3.10 and show that the attractor of the IFS is indeed the Cantor Set.

Example 3.15 Consider the IFS

$$\left\{[0,1]; \omega_1(x) = \frac{1}{3}x, \omega_2(x) = \frac{1}{3}x + \frac{2}{3}\right\}.$$

1. Show that this is an IFS with contraction factor $k = \frac{1}{3}$.
2. Let $B_0 = [0,1]$. Calculate $B_n = W^n(B_0)$, $n = 1, 2, 3, \ldots$ Deduce that $\mathcal{C} = \lim_{n\to\infty} B_n$ is the classical Cantor Set.

Solution:

1. We need to compute the contraction factor for each ω_i. Start with ω_1.

$$\begin{aligned} d(f(x), f(y)) &= d\left(\frac{1}{3}x, \frac{1}{3}y\right) \\ &= \left|\frac{1}{3}x - \frac{1}{3}y\right| \\ &= \left|\frac{1}{3}(x-y)\right| \\ &= \frac{1}{3}|x-y| \\ &= \frac{1}{3}d(x,y)\left|\frac{1}{3}x - \frac{1}{3}y\right|. \end{aligned}$$

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[18] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 195.

So $k_1 = \frac{1}{3}$. Now compute the contraction factor for ω_2.

$$\begin{aligned} d(f(x), f(y)) &= d\left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3}\right) \\ &= \left|\left(\frac{1}{3}x + \frac{2}{3}\right) - \left(\frac{1}{3}y + \frac{2}{3}\right)\right| \\ &= \left|\frac{1}{3}x - \frac{1}{3}y\right| \\ &= \left|\frac{1}{3}(x-y)\right| \\ &= \frac{1}{3}|x-y| \\ &= \frac{1}{3}d(x,y). \end{aligned}$$

So $k_2 = \frac{1}{3}$. Therefore, the contraction factor of the IFS is $\frac{1}{3}$.

2. To prove that the attractor of the IFS is the Cantor Set, we're going to prove that the action $W(B_n)$ is to remove the middle third of each interval in B_n. Proceed by induction on the iterates of the IFS on $B_0 = [0,1]$. Assume $[a,b]$ is a component of B_k with $k < n$.Then

$$\left[a, a + \frac{1}{3}(b-a)\right] \cup \left[b - \frac{1}{3}(b-a), b\right] \subset B_{k+1}$$

and

$$(a + \frac{1}{3}, b - \frac{1}{3}(b-a)) \cap B_{k+1} = .$$

Since $B_0 = [0,1]$,

$$B_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1].$$

In this case we are certainly removing the middle third of B_0 and the induction hypothesis is satisfied. Assume there is a component $[a', b']$

in B_{k-1} such that $[a, b]$ is the image of $[a', b']$ under some ω_i. By the induction hypothesis, the component $[a', b']$ had its middle third removed in B_n. Under the mapping ω_i, $a' \mapsto a$, $b' \mapsto b$ and $(b' - a') \mapsto (b - a)$. So $[a, b]$ is replaced in the set by

$$\begin{aligned} &\omega_i(\left[a', a' + \frac{1}{3}(b' - a')\right] \cup \left[b' - \frac{1}{3}(b' - a')\right]) \\ &= \left[a, a + \frac{1}{3}(b - a)\right] \cup \left[b - \frac{1}{3}(b - a)\right] \end{aligned}$$

Therefore, the operation is the removal of the middle third of $[a, b]$. Therefore, the attractor is the Cantor Set.

Now we have an idea of how an iterated function system works and how the attractor of an IFS is defines. We can begin to consider, informally, the concept of the address of a point in the attractor.

Let's begin by looking at the IFS from Example 3.15 $\left\{[0, 1]; \frac{1}{3}x, \frac{1}{3}x + \frac{2}{3}\right\}$. Now we know that, indeed, the Cantor Set is the attractor of the IFS. Notice that the Cantor Set is made up of two disjoint pieces, namely $\omega_1(\mathcal{C})$ and $\omega_2(\mathcal{C})$. But $\omega_1(\mathcal{C})$ and $\omega_2(\mathcal{C})$ also consist of two disjoint pieces, namely $\omega_1(\omega_1(\mathcal{C}))$, $\omega_2(\omega_1(\mathcal{C}))$, $\omega_1(\omega_2(\mathcal{C}))$, and $\omega_2(\omega_2(\mathcal{C}))$. This observation leads us to the notion of addressing the points in the attractor based on the sequence of contractions applied to $\mathcal{C}$. So the points that are part of $\omega_1(\omega_1(\mathcal{C}))$ will begin with a 11... and those that are part of $\omega_2(\omega_1(\mathcal{C}))$ will begin with 12... In fact, we can see that 0 will have the address $\bar{1}$ because we apply the contraction ω_1 infinitely many times to the Cantor set to get the point 0. Likewise, 1 will have the address $\bar{2}$. So we can find the addresses of points, but to establish this idea more formally we will need a new metric space called sequence space.

4 Shift Dynamical System

To establish more formally the idea of addresses we must first examine the set of all sequences of a given number of symbols. This set will turn out to be the set which contains all the addresses of points in the attractor of an IFS and it is called the sequence space or code space.

4.1 Code space and the defined distance metric

Code space is formally defined as follows for a given number of symbols, N.

Definition 4.1 The set of all infinite sequences of $1's$, $2's$...,and $N's$ is called the *sequence space* or *code space* of 1,...,N and is denoted by Σ_N. More precisely, $\Sigma_N = \{(s_1, s_2, s_3...) : s_i \in \{1, 2, 3..., N\}$ for all $i\}$.[19]

The code space associated with a given IFS $\{X; \omega_1, \omega_2, ...\omega_N\}$ will be Σ_N. We can defined following distance metric on Σ_N.

Definition 4.2 Let $\alpha, \beta \in \Sigma_N$. d_c is given by the following.

$$d_c(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{|\alpha_n - \beta_n|}{(N+1)^n} \text{ for all } \alpha, \beta \in \Sigma_N.^{20}$$

It can be shown that (Σ_N, d_c) is a metric space for a given N and that it is compact. There is a function that we are able to define on code space called the shift map. The shift map is defined as follows

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[19] Holmgren, Richard A. *A first course in discrete dynamical system*, second edition. 1996. p. 109.

[20] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 122.

Definition 4.3 The shift map $s : \Sigma_N \to \Sigma_N$ is defined by

$$s(\alpha_1, \alpha_2, ...) = \alpha_2, \alpha_3, ...$$

In other words, the shift map removes the first digit of the sequence. For example, let $0, 1, 2, 2, 1, 0, 1, ... \in \Sigma_2$, then $s(0, 1, 2, 2, 1, 0, 1, ...) = 1, 2, 2, 1, 0, 1, ...$[21]

This mapping will become important later on and we will eventually see that it is closely linked to inverses of transformations.

4.2 Addressing the points in the attractor

To formally define the notion of addresses requires the existence of a continuous function that maps points of the code space of N symbols onto the attractor of an IFS of N contractions. To prove the existence of such a function is very difficult and not very enlightening, but the theorems and proofs are included in Section 7.3 of the appendix for the interested reader. For the purposes of this paper is it sufficient to assert the existence of such a function and we'll denote this function as ϕ.

Definition 4.4 Let $\{\mathbf{X}; \omega_n : n = 0, 1, 2, ..., N\}$ be an IFS with associated code space Σ_N and A be the attractor of the IFS. Let $\phi : \Sigma_N \to A$ be the continuous function from code space onto the attractor of the IFS. An address of a point $a \in A$ is any member of the set

$$\{\sigma \in \Sigma : \phi(\sigma) = a\}.$$

---

[21] Holmgren, Richard A. *A first course in discrete dynamical system*, second edition. 1996. p. 114.

this set is called the set of addresses of $a \in A$.

Notice that nowhere in this definition are we restricted to having just one address for a given point. In the Cantor Set we noticed that the images of the Cantor Set under the contractions ω_1 and ω_2 were totally disjoint this means that for every point in the Cantor Set there is only one address. Attractors of this type are called totally disconnected and it is formally defined as follows.

Definition 4.5 The IFS is said to be *totally disconnected* if each point of its attractor possesses a unique address.[22]

But of course this is not the only type of attractor.

Example 4.6 Consider the IFS

$$\{[0,1]; \omega_1 = \frac{1}{2}x, \omega_2 = \frac{1}{2}x + \frac{1}{2}\}.$$

Notice first that the interval $[0,1]$ is the attractor of this IFS. Second, notice that the point $\frac{1}{2}$ is the image of two points; $\omega_1(1) = \frac{1}{2}$ and $\omega_2(0) = \frac{1}{2}$. This attractor is different from the Cantor Set because the images of the contractions are not disjoint, but they only overlap by one point. This attractor is classified as just-touching.

The property, just-touching, is formally defined as follows.

Definition 4.7 The IFS is said to be *just-touching* if it is not totally disconnected yet its attractor contains an open set $\mathcal{O}$ such that

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[22] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 125.

1. $\omega_i(\mathcal{O}) \cap \omega_\ell(\mathcal{O}) = \emptyset$ for all $i,\ell \in \mathbb{N}$.
2. $\bigcup_{i=1}^{N} \omega_i(\mathcal{O}) \subset \mathcal{O}$.

An IFS whose attractor obeys 1 and 2 is said to obey the open set conditions.[23]

Now we can show that the attractor of the IFS in Example 4.6 is just-touching.

Example 4.8 The attractor from Example 4.6 is not totally disconnected because $\omega_1(0) = \omega_2(1)$. Now we must show that it contains an open set that satisfies the open set conditions. Consider the open set $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

1. $$\begin{aligned}&\omega_1\left(\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right)\right)\cap\omega_2\left(\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right)\right)\\&=\left(0,\frac{1}{4}\right)\cup\left(\frac{1}{4},\frac{1}{2}\right)\cap\left(\frac{1}{2},\frac{3}{4}\right)\cup\left(\frac{3}{4},1\right)\\&=\emptyset\end{aligned}$$

   Thus condition 1 is satisfied.

2. $$\begin{aligned}\bigcup_{i=1}^{N}\omega_i\left(\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right)\right) &= \omega_1\left(\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right)\right)\cup\omega_2\left(\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right)\right)\\&=\left(0,\frac{1}{4}\right)\cup\left(\frac{1}{4},\frac{1}{2}\right)\cap\left(\frac{1}{2},\frac{3}{4}\right)\cup\left(\frac{3}{4},1\right)\\&\subset\left(0,\frac{1}{2}\right)\cup\left(\frac{1}{2},1\right)\end{aligned}$$

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[23] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 125.

Thus condition 2 is satisfied.

Therefore, the attractor is not totally disconnected and there exists an open set in the attractor that obeys the open set conditions. Thus the attractor is just-touching.

There is yet one more classification that takes care of all the other attractors. An attractor is said to be overlapping if it is neither just-touching nor disconnected.[24]

Definition 4.9 An IFS with some attractor A is said to be overlapping if for some ω_i, ω_j there exists an open set $\mathcal{P}$ such that

$$\mathcal{P} \subset \omega_i(A) \cap \omega_j(A)$$

An example of an overlapping IFS follows.

Example 4.10 Consider the IFS

$$\{[0,1]; \omega_1 = \frac{1}{2}x, \omega_2 = \frac{3}{4}x + \frac{1}{4}\}.$$

The attractor for the IFS is the line segment $[0, 1]$.

$$\begin{aligned} \omega_1([0,1]) &= \left[0, \frac{1}{2}\right] \\ \omega_2([0,1]) &= \left[\frac{1}{4}, 1\right] \end{aligned}$$

Therefore,

$$\left(\frac{1}{4}, \frac{1}{2}\right) \subset \left[\frac{1}{4}, \frac{1}{2}\right] = \omega_1([0,1]) \cap \omega_2([0,1])$$

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[24] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 125.

Thus there exists an open set in the intersection of two images. Thus the IFS is overlapping.

Now that we know how to address points, we can look at how applying a contraction of the IFS to a point changes the address. We will be able to define a dynamical system on the code space associated with IFS.This dynamical system will be called the shift dynamical system.

4.3 Shift Dynamical Systems

A dynamical system is defined as follows.

Definition 4.11 A *dynamical system* is a function $f : \mathbf{X} \to \mathbf{X}$ on a metric space $(\mathbf{X}, d)$. It is denoted by $\{\mathbf{X}; f\}$. The orbit of a point $x \in \mathbf{X}$ is the sequence $\{f^n(x)\}_{n=0}^{\infty}$

So, for example, $\{\Sigma_N; s\}$ where Σ_N is a code space and s is the shift map is a dynamical system. We can define a very similar dynamical system on the attractor of a totally disconnected IFS.

Definition 4.12 Let $\{\mathbf{X}; \omega_n : n = 0, 1, 2, ..., N\}$ be a totally disconnected IFS with attractor A. The *associated shift transformation* on A is the function $S : A \to A$ defined by

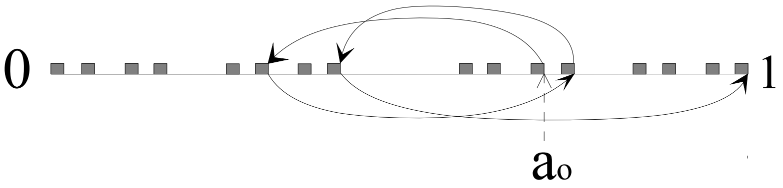
$$S(a) = \omega_n^{-1}(a) \text{ for } a \in \omega_n(A),$$

where ω_n is viewed as a function on A making it one-to one and onto. The dynamical system $\{A; S\}$ is called the *shift dynamical system* associated with the IFS.[25]

Example 4.13 Consider the once again the IFS

$$\left\{[0,1]; \frac{1}{3}x, \frac{1}{3}x + \frac{2}{3}\right\}.$$

This figure shows an approximation to the Cantor Set and an eventually periodic point $a_0 = \frac{61}{81} = \phi(2121\bar{2})$.



The orbit of a_0, $\{a_n = S^n(a_0)\}_{n=0}^{\infty}$, for the associated shift dynamical system is also shown. This orbit actually ends up at the fixed point $\phi(\bar{2}) = 1$. The orbit is $a_0 = \frac{61}{81} = \phi(2121\bar{2})$, $a_1 = \frac{7}{27} = \phi(121\bar{2})$, $a_2 = \frac{1}{9} = \phi(21\bar{2})$, $a_3 = \frac{1}{3} = \phi(1\bar{2})$, $a_4 = 1 = \phi(\bar{2})$, where $\phi : \sigma \to A$ is the associated code space map.

In the previous example notice the very close relationship between the shift map on code space and the shift dynamical system on the attractor. At this point the shift dynamical system is only defined for attractors of totally disconnected IFS's. What happens when the IFS is not totally disconnected? In this case $S(x)$ is not well defined for some $x \in A$. So we just decide to consider only one of the inverse transformations. We make the following definition for the case where there are only two transformations in the IFS.

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[25] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 140.

Definition 4.14 Let $\{\mathbf{X}; \omega_n : n = 1, 2\}$ be an IFS. Let A denote the attractor of the IFS. Assume that both $\omega_1 : A \to A$ and $\omega_2 : A \to A$ are invertible. A sequence of points $\{x_n\}_{n=0}^{\infty}$ in A is called an orbit of the *random shift dynamical system* associated with the IFS if

$$x_{n+1} = \begin{cases} \omega_1^{-1}(x_n) & \text{when } x_n \in \omega_1(A) \text{ and } x_n \notin \omega_1(A) \cap \omega_2(A), \\ \omega_2^{-1} & \text{when } x_n \in \omega_2(A) \text{ and } x_n \notin \omega_1(A) \cap \omega_2(A), \\ \text{one of } \{\omega_1^{-1}(x_n), \omega_2^{-1}(x_n)\} & \text{when } x_n \in \omega_1(A) \cap \omega_2(A), \end{cases}$$

for each $n \in \mathbb{N} \cup \{0\}$. We will use the notation $x_{n+1} = S(x_n)$ although there may be no well-defined transformation $S : A \to A$ that makes this true. Also we will write $\{A; S\}$ to denote the collection of possible orbits defined here, and we will call $\{A; S\}$ the random shift dynamical system associated with the IFS.[26]

Example 4.15 Consider the IFS from Example 4.6,

$$\{[0, 1]; \omega_1 = \frac{1}{2}x, \omega_2 = \frac{1}{2}x + \frac{1}{2}\}.$$

and the eventually fixed point $x_0 = \frac{5}{8} = 212\bar{1}$. Define the random shift dynamical system to be

$$x_{n+1} = \begin{cases} \omega_1^{-1}(x_n) & \text{when } x_n \in \omega_1(A) \text{ and } x_n \notin \omega_1(A) \cap \omega_2(A), \\ \omega_2^{-1} & \text{when } x_n \in \omega_2(A) \text{ and } x_n \notin \omega_1(A) \cap \omega_2(A), \\ \omega_1^{-1}(x_n) & \text{when } x_n \in \omega_1(A) \cap \omega_2(A), \end{cases}$$

This figure shows the attractor of the IFS, the eventually fixed point x_0, the point where the images of the contractions overlap, and the orbit of x_0 under the random shift dynamical system.

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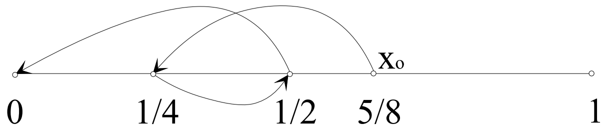
[26] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 153.



This orbit ends up at the fixed point $\phi(\bar{1}) = 0$. The orbit is $x_0 = \frac{5}{8} = \phi(212\bar{1})$, $x_1 = \frac{1}{4} = \phi(12\bar{1})$, $x_2 = \frac{1}{2} = \phi(2\bar{1})$, $x_3 = 0 = \phi(\bar{2})$, where $\phi : \sigma \to A$ is the associated code space map.

Now that we understand addressing points and the shift dynamical systems, we have all the tools necessary to understand the Shadowing Theorem.

5 Shadowing Theorem

The shadowing theorem says is that if we compute the orbit of a point x under the associated shift or random shift dynamical system $\{A; S\}$ that has error introduced, then there will exist a point x_0 whose orbit is within a certain distance of the orbit of x. That distance depends on the contraction factor of the IFS and the error introduced.

Theorem 5.1 The Shadowing Theorem

Let $\{\mathbf{X}; \omega_n : n = 0, 1, 2, ..., N\}$ be an IFS with of contraction k, where $0 < k < 1$. Let A denote the attractor of the IFS and suppose that each of the transformations $\omega_n : A \to A$ is invertible. Let $\{A; S\}$ denote the associated shift dynamical system in the case that the IFS is totally disconnected; otherwise let $\{A; S\}$ denote the associated random shift dynamical system.

Let $\{\tilde{x}_n\}_{n=0}^{\infty} \subset A$ be an approximate orbit of S, such that

$$d(\tilde{x}_{n+1}, S(\tilde{x}_n)) \leq \theta \text{ for all } n \in \{\mathbb{N} \cup 0\}$$

for some fixed constant θ with $0 \leq \theta \leq diam(A)$. Then there is an exact orbit $\{x_n = S^n(x_0)\}_{n=0}^{\infty}$ for some $x_0 \in A$, such that

$$d(\tilde{x}_{n+1}, x_{n+1}) \leq \frac{k\theta}{(1-k)} \text{ for all } n \in \{\mathbb{N} \cup 0\}.$$

This orbit is called the shadowing orbit.

Proof:

For $n \in \mathbb{N}$, let $\sigma_n \in \{1, 2, ..., N\}$ be chosen so that $\omega_{\sigma_1}^{-1}, \omega_{\sigma_2}^{-1}, \omega_{\sigma_2}^{-1}, ...$ is the actual sequence of inverse maps used to compute $S(\tilde{x}_0), S(\tilde{x}_1), S(\tilde{x}_2), ...$ Let $\phi : \Sigma \rightarrow A$ denote the code space map associated with the IFS. Then define

$$x_0 = \phi(\sigma_1, \sigma_2, \sigma_3 ...).$$

Then we compare the exact orbit of the point x_0,

$$\{x_n = S^n(x_0) = \phi(\sigma_{n+1}, \sigma_{n+2}, \sigma_{n+3} ...)\}_{n=0}^{\infty}$$

with the errorful orbit $\{\tilde{x}_n\}_{n=0}^{\infty}$.

Let M be a large positive integer. Then, since x_M and $S(\tilde{x}_{M-1})$ both belong to A, we have

$$d(S(x_{M-1}), S(\tilde{x}_{M-1})) \leq \text{diam}(A) < \infty.$$

Since $S(x_{M-1})$ and $S(\tilde{x}_{M-1})$ are both computed with the same inverse map $\omega_{\sigma_M}^{-1}$ it follows that

$$d(x_{M-1}, \tilde{x}_{M-1}) \leq s \text{ diam}(A).$$

Hence

$$\begin{aligned} d(S(x_{M-2}), S(\tilde{x}_{M-2})) &= d(x_{M-1}, S(\tilde{x}_{M-2})) \\ &\leq d(x_{M-1}, \tilde{x}_{M-1}) + d(\tilde{x}_{M-1}, S(\tilde{x}_{M-2})) \\ &= \theta + s \text{ diam}(A); \end{aligned}$$

and repeating the argument used above we now find

$$d(x_{M-2}, \tilde{x}_{M-2}) \leq s(\theta + s\text{diam}(A)).$$

Repeating the same argument k time we arrive at

$$d(x_{M-k}, \tilde{x}_{M-k}) \leq s\theta + s\theta^2 + ... + s^{k-1} + s^k \text{ diam}(A).$$

Hence for any positive integer M and any integer n such that $0 < n < M$, we have

$$d(x_n, \tilde{x}_n) \leq s\theta + s\theta^2 + ... + s^{M-n-1} + s^{M-n} \text{ diam}(A).$$

Now take the limit of both sides of the equation as $M \rightarrow \infty$ to obtain

$$d(x_n, \tilde{x}_n) \leq s\theta(1 + k + k2 + ...) = \frac{k\theta}{(1-k)}, \text{ for all } n \in \mathbb{N}$$

This completes the proof.[27] ▮

The Shadowing Theorem guarantees the existence of a point whose orbit under the shift dynamical system stays within $\frac{k\theta}{(1-k)}$ of our errorful orbit where k is the contraction factor for the IFS and θ is the error that we introduce.

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[27] Taken from Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 160.

6 Conclusion

So it is true! We can calculate an errorful orbit and there will exist an exact orbit of another point, a shodowing orbit, within a certain distance from the errorful orbit. So we need not be worried about the limitations of our calculators and computers, or do we...Does the calculator example satisfy the conditions of the hypothesis? The action of dividing any real number by three can be represented by the following IFS.

$$\left\{\mathbb{R}\ ; \frac{1}{3}x\right\}$$

The attractor of this IFS is the set containing zero and the Shadowing Theorem states that the errorful orbit $\{\tilde{x}_n\}_{n=0}^{\infty}$ must be a subset of the attractor, $\{0\}$. Notice that the errorful orbit computed by the calculator is not a subset of $\{0\}$. So our example does not satisfy the condidtions of the hypothesis, and therefore, the Shadowing Theorem does not guarantee that there is an exact orbit within a certain distance from our errorful one. While the Shadowing Theorem does not apply to our example, the intuition gained by considering the example is useful for seeing how the theorem works. Suppose we try to construct an example from the IFS

$$\left\{[0,1]; \frac{1}{3}x, \frac{1}{3}x+\frac{2}{3}\right\}.$$

We first need an errorful orbit that is a subset of the attractor, the Cantor Set, $\mathcal{C}$. Finding an orbit of a given point in the Cantor Set is not a problem, but how do we establish that the errorful orbit of that point is in the Cantor

Set? This, in itself, is not an easy task. However, assuming we have done this, how then do we find the exact orbit that shadows it? The Shadowing Theorem is powerful enough to establish the existence of such an orbit, but it does not describe how to construct the orbit. One possible approach to completeing this task would be to define the function $\phi : \Sigma_2 \to \mathcal{C}$ from Section 4.2.Then we could work in the symbol space and translate our work to the Cantor Set, but such an approach is beyond the scope of this project. Thus with the Shadowing Theorem we are able to assert the existence of a shadowing orbit, but not construct the shadowing orbit.

7 Appendix

7.1 Metric spaces

A metric space defined as follows.

Definition 7.1 Let $\mathbf{X}$ be a set. Let $d : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ be a real-valued function with the following properties:

Positive- For all $a, b \in X$, $d(a, b) \geq 04$.

Positive definite- For $a, b \in \mathbf{X}$, $d(a, b) = 0$ if and only if $a = b$.

Symmetric- For all $a, b \in \mathbf{X}$, $d(a, b) = d(b, a)$.

Triangle inequality- For all $a, b, c \in \mathbf{X}$,

$$d(a, c) \leq d(a, b) + d(b, c).$$

The function d is called the *distance function* or *metric* for $\mathbf{X}$. The set $\mathbf{X}$ together with d is called a *metric space* and is denoted by $(\mathbf{X}, d)$. The elements of $\mathbf{X}$ are called *points* of the metric space. [28]

Theorem 7.2 Let $(\mathbf{X}, d)$ be a complete metric space, then $(\mathcal{H}(\mathbf{X}), h)$ is a metric space.

Proof:

Let $(\mathbf{X}, d)$ be a metric space and $A, B, C \in \mathcal{H}(\mathbf{X})$. Show the following properties hold for h.

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[28] Schumacher, Carol S. *The Analysis Tree*, Spring 1999 version. p. 26-27.

Positive- $h(A,B) = \max\{d(A,B), d(B,A)\}$. Since both $d(A,B)$ and $d(B,A)$ are non-negative real numbers the maximum of $d(A,B)$ and $d(B,A)$ will be either $d(A,B)$ or $d(B,A)$, $h(A,B)$ is greater than or equal to zero.

Positive definite-

($\Rightarrow$) Suppose $h(A,B) = 0$. Let $a \in A$. Since $h(A,B) = 0$, both $d(A,B)$ and $d(B,A)$ equal zero because d is always positive and real. Since $d(A,B) = 0$, $d(a,B) = 0$ for $a \in A$. Remember that$d(a,B) = \min\{d(a,y) : y \in B\}$, so there is a $y \in B$ such that $d(a,y) = 0$. Since d is a distance metric and $d(a,y) = 0$, $a = y$. Therefore, $a \in B$ and $A \subseteq B$. By a parallel argument $B \subseteq A$. Therefore, $A = B$.

($\Leftarrow$) Suppose $A = B$. We must show that both $d(A,B)$ and $d(B,A)$ equal 0. Since $A = B$, $d(A,B) = d(A,A) = d(B,A)$. Note that $d(a,A) = 0$ for all $a \in A$. So $d(A,A) = 0$. Therefore, $d(A,B) = 0$ and $d(B,A) = 0$, so $h(A,B) = 0$.

Symmetric- Since $\max\{d(A,B), d(B,A)\} = \max\{d(B,A), d(A,B)\}$, $h(A,B) = \max\{d(A,B), d(B,A)\} = \max\{d(B,A), d(A,B)\} = h(B,A)$.

Triangle inequality- To show that $h(A,B) \leq h(A,C) + h(C,B)$ it is sufficient to show that $d(A,B) \leq d(A,C) + d(C,B)$ and that $d(B,A) \leq d(B,C) + d(C,A)$, for then we will have

$$h(A,B) \quad = \quad \max\{d(A,B), d(B,A)\}$$

$$
\begin{aligned}
&\leq \max\{d(A,C)+d(C,B), d(B,C)+d(C,A)\} \\
&\leq \max\{d(A,C), d(C,A)\} + \max\{d(B,C), d(C,B)\} \\
&= h(A,C)+h(C,B)
\end{aligned}
$$

Let's start with showing $d(a,B) \leq d(a,C)+d(C,B)\ \forall a \in A$. Let $a \in A$. We know that

$$
\begin{aligned}
d(a,B) &= \min\{d(a,b) : b \in B\} \\
&\leq \min\{d(a,c)+d(c,b) : b \in B\} \forall c \in C \\
&= d(a,c)+\min\{d(c,b) : b \in B\} \forall c \in C
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d(a,B) &\leq \min\{d(a,c) : c \in C\} + \max\{\min\{d(c,b) : b \in B\} : c \in C\} \\
&= d(a,C)+d(C,B).
\end{aligned}
$$

This is true for all $a \in A$,so

$$d(A,B) \leq d(A,C)+d(C,B)$$

By a parallel argument we can show that

$$d(B,A) \leq d(B,C)+d(C,A)$$

Therefore,$h(A,B) \leq h(B,C)+h(A,C)$. Thus the triangle equality is satisfied.[29]

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[29] The proof of the triangle inequality is taken from Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 33.

Thus $(\mathcal{H}(\mathbf{X}), h)$ is a metric space. ▮

In these metric spaces we can define sequences. Sometimes the sequences will converge and sometimes they won't. We often times think of convergent sequences as those sequences whose consecutive terms get closer and closer together, but it is not always true that such sequences converge. These types of sequences are called Cauchy sequences and are formally defined as follows.

Definition 7.3 Let (a_n) be a sequence in a metric space $\mathbf{X}$. Then (a_n) is said to be *Cauchy* provided that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(a_n, a_m) < \epsilon$ whenever $n, m > N$.[30]

Definition 7.4 A metric space (X, d) is said to ba complete if every Cauchy sequence in X converges.[31]

Definition 7.5 Let (X, d) be a metric space. Let $U \subseteq X$. The we say that U is an open subset or (if X is understood) simply an open set, if for every $a \in U$ there exists $r > 0$ such that the set $\{x \in U : d(a, x) < r\}$ is totally contained within U.[32]

Definition 7.6 Let X be a metric space, and $S \subseteq X$. If $\{U_\alpha\}_{\alpha \in \Lambda}$ is a collection of open subsets of X, and

$$S \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$$

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[30] Schumacher, Carol S. *The Analysis Tree*, Spring 1999 version. p. 66.

[31] Schumacher, Carol S. *The Analysis Tree*, Spring 1999 version. p. 69.

[32] Schumacher, Carol S. *The Analysis Tree*, Spring 1999 version. p. 33.

we say that $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover for S. Any subcollection of $\{U_\alpha\}_{\alpha\in\Lambda}$ whose union still contains S is called a subcover for S.[33]

Definition 7.7 Let X be a metric space. A subset S of X is said to be *compact* provided that every open cover for S has a finite subcover.[34]

Theorem 7.8 Let $(\mathbf{X}, d)$ be a complete metric space. Let $S \subset \mathbf{X}$. Then S is compact if and only if it is closed and totally bounded.[35]

Definition 7.9 Let S be a subset of a metric space $(\mathbf{X}, d)$. S is totally bounded if, for each $\epsilon > 0$, there is a finite set of points $\{y_1, y_2, ..., y_n\} \subset S$ such that if $x \in \mathbf{X}$, $d(x, y_i) < \epsilon$ for some $y_i \in \{y_1, y_2, ..., y_n\}$. This set points, $\{y_1, y_2, ..., y_n\}$, is called an *ϵ-net*.[36]

Lemma 7.10 Let $(\mathbf{X}, d)$ be a metric space. Let $\{A_n\}_{n=1}^{\infty}$ be a Cauchy sequence of points in $(\mathcal{H}(\mathbf{X}), h)$. Let $\{n_j\}_{j=1}^{\infty}$ be an infinite sequence of integers

$$0 < n_1 < n_2 < n_3 < ...$$

Suppose that we have a Cauchy sequence $\{x_{n_j}\}_{n=1}^{\infty}$ such that $x_{n_k} \in A_{n_k}$ in $(\mathbf{X}, d)$. Then there is a Cauchy sequence $\{\tilde{x}_n\}_{n=1}^{\infty}$ such that $\tilde{x}_n \in A_n$ and $\tilde{x}_{n_j} = x_{n_j}$, for all $j \in \mathbb{N}$ and $\{x_n\}$. We say that $\{\tilde{x}_n\}$ is an extension of $\{x_{n_j}\}$ to $\{A_n\}$.

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[33] Schumacher, Carol S. *The Analysis Tree*, Spring 1999 version. p. 71.

[34] Schumacher, Carol S. *The Analysis Tree*, Spring 1999 version. p. 72.

[35] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 20.

[36] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 20.

Proof:

We are going to construct the sequence $\{\tilde{x}_n\}$. For each $n \in \{1, 2, ..., n_1\}$, choose $\tilde{x}_n \in \{x \in A_n | d(x, x_{n_1}) = d(x_{n_1}, A_n\}$. That is, $\tilde{x}_n$ is the a point in A_n that is closest to x_{n_1} which is in A_{n_1}. Since A_n is compact, we know that such a point exists. We repeat this process for each $j \in \{2, 3, ...\}$ and each $n \in \{n_j + 1, n_j + 2, ...n_{j+1}\}$ by choosing $\tilde{x}_n \in \{x \in A_n | d(x, x_{n_{j+1}}) = d(x_{n_{j+1}}, A_n\}$.

To show that $\{\tilde{x}_n\}$ is an extension of $\{x_{n_j}\}$ to $\{A_n\}$ we must show the following:

1. $\{\tilde{x}_n\}$ is a Cauchy sequence.

   Let $\epsilon > 0$. Since $\{x_{n_j}\}$ we know there exists an $N_1 \in \mathbb{N}$ such that

   $$d(x_{n_k}, x_{n_j}) \leq \frac{\epsilon}{3}.$$

   Also, since $\{A_n\}$ is Cauchy, there exists N_2 such that

   $$d(A_m, A_n) \leq \frac{\epsilon}{3}.$$

   Let $N = \max(N_1, N_2)$. By the triangle inequality, for $m, n \geq N$,

   $$d(\tilde{x}_m, \tilde{x}_n) \leq d(\tilde{x}_m, x_{n_j}) + d(x_{n_j}, x_{n_k}) + d(x_{n_k}, \tilde{x}_n),$$

   where $m \in \{n_{j-1}+1, n_{j-1}+1, ..., n_j\}$ and $n \in \{n_{k-1}+1, n_{k-1}+1, ..., n_k\}$. Since

   $$h(A_m, A_{n_j}) < \frac{\epsilon}{3},$$

   We know that the following is true.

   $$A_m \cap \left(\{x_{n_j}\} + \frac{\epsilon}{3}\right) \neq \emptyset$$

Therefore, $d(\tilde{x}_m, x_{n_j}) \leq \frac{\epsilon}{3}$. Likewise, $d(x_{n_k}, \tilde{x}_n) \leq \frac{\epsilon}{3}$. Thus $d(\tilde{x}_m, \tilde{x}_n) \leq \epsilon$. for all $m, n > N$. Thus $\{\tilde{x}_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

2. $\tilde{x}_n \in A_n$ for all $n \in \mathbb{N}$.

   This follows directly from the construction of $\{\tilde{x}_n\}_{n=1}^{\infty}$.

3. $\tilde{x}_{n_j} = x_{n_j}$, for all $j \in \mathbb{N}$.

   This follows directly from the construction of $\{\tilde{x}_n\}_{n=1}^{\infty}$.

Thus $\{\tilde{x}_n\}$ is an extension of $\{x_{n_j}\}$ to $\{A_n\}$.[37] ∎

Theorem 7.11 Let $(\mathbf{X}, d)$ be a complete metric space. Then $(\mathcal{H}(\mathbf{X}), h)$ is a complete metric space. Moreover, if $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}(\mathbf{X})$, then

$$A = \lim_{n \to \infty} A_n$$

can be characterized as follows:

$$A = \left\{ x \in \mathbf{X} : x = \lim_{n \to \infty} x_n \text{ where } \{x_n\}_{n=0}^{\infty} \text{ is a Cauchy sequence such that } x_n \in A_n \right\}$$

Proof:

Let $\{A_n\}$ be a Cauchy sequence in $\mathcal{H}(\mathbf{X})$ and let A be defined as above. Now we must show that the following hold.

1. $A \neq \emptyset$.

   We shall prove that this is true by constructing a Cauchy sequence

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[37] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 34-35.

$\{a_i\}$ such that $a_i \in A_i$ in $\mathbf{X}$. We begin by finding a strictly increasing sequence $\{N_n\}$ such that

$$h(A_m, A_n) < \frac{1}{2^i} \text{ for } m, n > N_i.$$

Proceed by induction on the N_i's. Choose $x_{N_i} \in A_{N_i}$. Then since

$$H(A_{N_1}, A_{N_2}) \leq \frac{1}{2^1},$$

we can find an $x_{N_2} \in A_{N_2}$ such that

$$d(x_{N_1}, x_{N_2}) \leq \frac{1}{2^1}.$$

Assume that we have chosen a finite sequence $\{x_{N_i}\}_{i=1}^{k}$, such that $x_{N_i} \in A_{N_i}$, for which

$$d(x_{N_{i-1}}, x_{N_i}) \leq \frac{1}{2^{i-1}}.$$

Then since $h(A_{N_k}, A_{N_{k-1}}) \leq \frac{1}{2^k}$, and $x_{N_k} \in A_{N_k}$, we can find $x_{N_{k+1}} \in A_{N_{k+1}}$ such that

$$d(x_{N_k}, x_{N_{k+1}}) \leq \frac{1}{2^k}.$$

Thus, by induction, we can find an infinite sequence $\{x_{N_i}\}$ such that $x_{N_i} \in A_{N_i}$ and

$$d(x_{N_i}, x_{N_{i+1}}) \leq \frac{1}{2_i}.$$

Now we must show that $\{x_{N_i}\}$ Cauchy. Let $\epsilon > 0$ and choose N_ϵ such that

$$\sum_{i=N_\epsilon}^{\infty} \frac{1}{2^i} < \epsilon.$$

then for $m > n \geq N_\epsilon$ we have

$$\begin{aligned} d(x_{N_m}, x_{N_n}) &\leq d(x_{N_m}, x_{N_{m+1}}) + d(x_{N_{m+1}}, x_{N_{m+2}}) + ... + d(x_{N_{n-1}}, x_{N_n}) \\ &< \sum_{i=N_\epsilon}^{\infty} \frac{1}{2^i} \\ &< \epsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy. Therefore, by Lemma 7.10 there exists a convergent subsequence $\{a_i\}$ such that $a_i \in A_i$ for which $a_{N_i} = x_{N_i}$. Then the limit of $\{a_i\}$ exists and by definition is in A. Thus $A \neq \emptyset$.

2. A is closed and hence complete since $\mathbf{X}$ is complete.

   To show that A is closed we suppose $\{a_i\}$ such that $a_i \in A_i$ is a sequence that converges to a point a, and show that $a \in A$. For each positive integer i, there exists a sequence $\{x_{i,n}\}_{n=1}^{\infty}$ such that $x_{i,n} \in A_n$ and $\lim_{n\to\infty} x_{i,n} = a_i$. By an argument parallel to the one above, there exists an increasing sequence of positive numbers $\{N_i\}_{i=1}^{\infty}$ such that

   $$d(a_{N_i}, a) < \frac{1}{i}.$$

   Also, by an argument parallel to the one above, there is a sequence of integers $\{m_i\}$ such that

   $$d(x_{N_i,m_i}, a_{N_i}) \leq \frac{1}{i}.$$

   Thus

   $$(x_{N_i,m_i}, a) \leq \frac{2}{i}.$$

   If we let $y_{m_i} = x_{N_i,m_i}$ we see that $y_i \in A_{m_i}$ and $\lim_{i\to\infty} y_{m_i} = a$. By Lemma 7.10, $\{y_{m_i}\}$ can be extended to a convergent sequence $\{z_i\}$ such that $z_i \in A_i$, and so $a \in A$. Thus A is closed.

3. For $\epsilon > 0$ there is N such that for $n \geq N$, $a \subset A_n + \epsilon$.

   Let $\epsilon > 0$. Then there exists and N such that for $m, n \geq N$, $h(A_m, A_n) \leq \epsilon$. Now let $n \geq N$. Then for $m \geq n$, $A_m \subset A_n + \epsilon$. Show that $A \subset A_n + \epsilon$. Let $a \in A$. There is a sequence $\{a_i\}$ such that $a_i \in A_i$ that converges to a. Assume N is large enough such that if $m \geq N$, $d(a_m, a) < \epsilon$. Then $a_m \in A_n + \epsilon$ since $A_m \subset A_n + \epsilon$. Since A_n is compact it is closed. Since $a_m \in A_n + \epsilon$ for all $m \geq N$, a must also be in $A_n + \epsilon$.

4. A is totally bounded and thus compact.

   Proceed by contradiction. Suppose A were not totally bounded. Then for some $\epsilon > 0$ there would not exists a finite ϵ-net[38]. So we could find a sequence $\{x_i\}$ in A such that $d(x_i, x_j) \geq \epsilon$ for $i \neq j$. By 3 there exists an n large enough so that $A \subset A_n + \frac{\epsilon}{3}$. For each x_i, there is a corresponding $y_i \in A_n$ for which $d(x_i, y_i) \leq \frac{\epsilon}{3}$. Since A_n is compact, some subsequence $\{y_{n_i}\}$ of $\{y_i\}$ converges. We can find two points y_{n_i} and y_{n_j} such that $d(y_{n_i}, y_{n_j}) \leq \frac{\epsilon}{3}$. But then

   $$\begin{aligned} d(x_{n_i}, x_{n_j}) &\leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, y_{n_j}) + d(y_{n_j}, x_{n_j}) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}, \end{aligned}$$

   This is a contradiction. Thus A is totally bounded. Therefore, we deduce that A is compact, because we have closure from 2.

5. $\lim_{n\to\infty} A_n = A$.

   From part 4, $A \in \mathcal{H}(\mathbf{X})$. From part 3 and Lemma 2.6 we know that it is

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[38] See Definition 7.9 in the Appendix

sufficient to show that there exists N such that for $n \geq N$, $A_n \subset A + \epsilon$. Let $\epsilon > 0$. Let N be such that for $m, n \geq N$, $h(A_m, A_n) \leq \frac{\epsilon}{2}$. Then for $m, n \geq N$, $A_m \subset A_n + \frac{\epsilon}{2}$. Let $n \geq N$. Show $A_n \subset A + \epsilon$. Let $y \in A_n$. There exists a strictly increasing sequence of integers greater than n, N_i, and for $m, n \geq N_j$, $A_m \subset A_n + \frac{\epsilon}{2^{j+1}}$. Note that $A_n \subset A_{N_1} + \frac{\epsilon}{2}$. Since $y \in A_n$, there is an $x_{N_1} \in A_{N_1}$ such that $d(y, x_{N_1}) \leq \frac{\epsilon}{2}$. Since $x_{N_1} \in A_{N_1}$, there is a point $x_{N_2} \in A_{N_2}$ such that $d(x_{N_1}, x_{N_2}) \leq \frac{\epsilon}{2^2}$. In a similar manner we can use induction to find a sequence $\{x_{N_n}\}_{n=1}^{\infty}$, such that $x_{N_j} \in A_{N_j}$ and $d(x_{N_j}, x_{N_{j+1}}) \leq \frac{\epsilon}{2^{j+1}}$. Using the triangle inequality we see that

$$d(y, x_{N_j}) \leq \frac{\epsilon}{2}$$

for all $j \in \mathbb{N}$, and that $\{x_{N_j}\}$ is a Cauchy sequence. Because of the way that n was chosen, each $A_{N_j} \subset A_n + \frac{\epsilon}{2}$. Since $\{x_{N_j}\}$ is a Cauchy sequence, it converges to a point x and since $A_n + \frac{\epsilon}{2}$ is closed, $x \in A_n + \frac{\epsilon}{2}$. Additionally, $d(y, x_{N_j}) \leq \epsilon$ implies that $d(x, y) \leq \epsilon$. Thus $A_n \subset A + \epsilon$ for $n \geq N$, so $\lim_{n\to\infty} A_n = A$.

Thus $(\mathcal{H}(\mathbf{X}), h)$ is a complete metric space. [39] ∎

7.2 Functions

Definition 7.12 A function $f : A \to B$ is said to be *one-to one* if given $b \in B$, there is at most one $a \in A$ for which $b = f(a)$. A function $f : A \to B$

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[39] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 36-37.

is said to be *onto* if for each $b \in B$, there is at least one $a \in A$ for which $b = f(a)$. [40]

Definition 7.13 If a function f is one-to-one and onto, it is invertible. We denote the inverse of f as f^{-1} and define $f^{-1} : \mathbf{X} \to \mathbf{X}$ such that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for all $x \in \mathbf{X}$. where $x \in \mathbf{X}$ is the unique point such that $y = f(x)$.[41]

7.3 Addresses of points

Definition 7.14 Let $(\mathbf{X}, d)$ be a metric space and let $C \in \mathcal{H}(\mathbf{X})$. Define a transformation $\omega_0 : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$ by $\omega_0(B) = C$ for all $B \in \mathcal{H}(\mathbf{X})$. Then ω_0 is called a *condensation transformation* and C is called the *condensation set.*[42]

Definition 7.15 Let $\{\mathbf{X}; \omega_n : n = 1, 2, ..., N\}$ be an IFS with contraction factor $0 \leq k < 1$. Let $\omega_0 : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$ be a condensation transformation. Then $\{\mathbf{X}; \omega_n : n = 0, 1, 2, ..., N\}$ is called an *IFS with condensation*, with contraction.[43]

Fact 7.16 A sequence of set $\{A_n \subset \mathbf{X}\}_{n=0}^{\infty}$, where $(\mathbf{X}, d)$ is a metric space, is said to be increasing if $A_0 \subset A_1 \subset A_2 \subset ...$ and decreasing if $A_0 \supset A_1 \supset A_2 \supset ....$ The inclusions are not necessarily strict. A decreasing

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[40] Schumacher, Carol S. *Chapter Zero*. 1996. p. 51.

[41] Schumacher, Carol S. *Chapter Zero*. 1996. p. 54.

[42] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 91.

[43] Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. p. 91.

sequence of sets $\{A_n \subset \mathcal{H}(\mathbf{X}\}_{n=0}^{\infty}$ is a Cauchy sequence. If $\mathbf{X}$ is compact, then an increasing sequence of sets $\{A_n \subset \mathcal{H}(\mathbf{X}\}_{n=0}^{\infty}$ is a Cauchy sequence. Let $\{\mathbf{X}; \omega_n : n = 0, 1, 2, ..., N\}$ be an IFS with condensation set C, and let $\mathbf{X}$ be compact. Let $W_0(B) = \bigcup_{n=0}^{N} \omega_n(B)$ for all $B \in \mathcal{H}(\mathbf{X}$ and let $W(B) = \bigcup_{n=1}^{N} \omega_n(B)$. Define $\{C_n = W_0^{\circ n}(C)\}_{n=0}^{\infty}$. Then C_n is a Cauchy sequence in $\mathcal{H}(\mathbf{X}$ that converges to the attractor of the IFS. Observe that

$$C_n = C \cup W(C) \cup W^2(C) \cup ... \cup W^n(C)$$

provides an increasing sequence of compact sets. It follows immediately that the limit set A obeys $W_0(A) = A$.

Lemma 7.17 Let $\mathbf{X}; \omega_n : n = 0, 1, 2, ...N$ be an IFS, where $(\mathbf{X}, d)$ is a metric space. Let $K \in \mathcal{H}(\mathbf{X})$. Then there exists $\tilde{K} \in \mathcal{H}(\mathbf{X})$ such that $K \subset \tilde{K}$ and $\omega_n : \tilde{K} \to \tilde{K}$ for $n = 1, 2, 3..N$. In other words, $\{\tilde{K}; \omega_n : n = 1, 2, 3...N\}$ is an IFS where the underlying space is compact.

Proof:

Define $W : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$ by

$$W(B) = \bigcup_{n=1}^{N} \omega_n(B) \text{ for all } B \in \mathcal{H}(\mathbf{X}).$$

To construct $\tilde{K}$ consider the IFS with condensation[44] $\{\mathbf{X}; \omega_n : n = 0, 1, 2, ..., N\}$, where the condensation map ω_0 is associated with the condensation set K. The attractor of this IFS belongs to $\mathcal{H}(\mathbf{X})$. By Fact 7.16 $\tilde{K}$ can be written as

$$\tilde{K} = \text{closure of}(K \cup W(K) \cup W^2(C) \cup ... \cup W^n(C) \cup ...).$$

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[44] See Definition 7.14 and Definition 7.15 in the appendix.

It is readily seen that $K \subset \tilde{K}$ and $W(\tilde{K}) \subset \tilde{K}$.[45] ▮

Lemma 7.18 Let $\{\mathbf{X}; \omega_n : n = 0, 1, 2, ..., N\}$ be an IFS of contraction factor s where $(\mathbf{X}, d)$ is a complete metric space. Let (Σ_N, d_c) denote the code space associated with the IFS. For each $\sigma \in \Sigma$, $n \in \mathbb{N}$, and $x \in \mathbf{X}$, define

$$\phi(\sigma, n, x) = \omega_{\sigma_1} \circ \omega_{\sigma_2} \circ ... \circ \omega_{\sigma_n}(x).$$

Let K denote a compact nonempty subset of $\mathbf{X}$. Then there is a real constant D such that

$$d(\phi(\sigma, m, x_1), \phi(\sigma, n, x_2)) \leq D s^{\min\{m,n\}}$$

for all $\sigma \in \Sigma_N$, all $m, n \in \mathbb{N}$ and all $x_1, x_2 \in K$.

Proof:

Let σ, m, n, x_1 and x_2 be as stated in the lemma. Construct $\tilde{K}$ from K as in Lemma 7.17. Without any loss of generality we can suppose that $m < n$. then observe that

$$\phi(\sigma, n, x_2) = \phi(\sigma, m, \phi(\rho, n - m, s_2)).$$

where

$$\rho = \sigma_{n-m+1}\sigma_{n-m+1}...\sigma_n... \in \Sigma_N$$

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[45] This theorem and proof are taken directly from Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. pp. 122-3. I intend to rewrite the proof for the final draft.

Let $x_3 = \phi(\rho, n-m, x_2)$. The x_3 belongs to $\tilde{K}$. Hence we can write

$$\begin{aligned} d(\phi(\sigma,m,x_1),\phi(\sigma,n,x_2)) &= d(\phi(\sigma,m,x_1),\phi(\sigma,m,x_3)) \\ &\leq sd(\omega_{\sigma_2}\circ ...\circ \omega_{\sigma_m}(x_1),\omega_{\sigma_2}\circ ...\circ \omega_{\sigma_m}(x_3)) \\ &\leq s^2 d(\omega_{\sigma_3}\circ ...\circ \omega_{\sigma_m}(x_1),\omega_{\sigma_3}\circ ...\circ \omega_{\sigma_m}(x_3)) \\ &\leq s^m d(x_1,x_3) \\ &\leq s^m D, \end{aligned}$$

where $D = max\{d(x_1,x_3) : x_1, x_3 \in \tilde{K}\}$. D is finite because $\tilde{K}$ is compact. This complete the proof. [46] ▮

Theorem 7.19 Let $\{\mathbf{X}; \omega_n : n = 0,1,2,...,N\}$ be an IFS of contraction factor s where $(\mathbf{X}, d)$ is a complete metric space. Let A denote the attractor of the IFS. Let (Σ_N, d_c) denote the code space associated with the IFS. For each $\sigma \in \Sigma$, $n \in \mathbb{N}$, and $x \in \mathbf{X}$, define

$$\phi(\sigma,n,x) = \omega_{\sigma_1}\circ\omega_{\sigma_2}\circ ...\circ\omega_{\sigma_n}(x).$$

Then

$$\phi(\sigma) = \lim_{n\to\infty}\phi(\sigma,n,x)$$

exists, belongs to A, and is independent of $x \in \mathbf{X}$. If K is a compact subset of X, then the convergence is uniform over $x \in K$. The function $\phi : \Sigma_N \to A$ thus provided is continuous and onto.

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[46] This theorem and proof are taken from Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. pp. 123. I intend to rewrite the proof of the final draft.

Proof:

Let $x \in \mathbf{X}$. Let $K \in \mathcal{H}(\mathbf{X})$ be such that $x \in K$. Construct $\tilde{K}$ as in Lemma 7.17. Define $W : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$ in the usual way. By Theorem 3.12, W is a contraction mapping on the space $(\mathcal{H}(\mathbf{X}), h(d))$; and we have

$$A = \lim_{n\to\infty} \{W^n(K)\}.$$

In particular $\{W^{\circ n}(K)\}$ is a Cauchy sequence in $\mathcal{H}(\mathbf{X})$. Notice that $\phi(\sigma, n, x) \in W^{\circ n}(K)$. It follows from Theorem 2.7 that if $\lim_{n\to\infty} \phi(\sigma, n, x)$ exists, then it belongs to A.

That the latter limit does exist follows from the fact that, for fixed $\sigma \in \Sigma_N$, $\{\phi(\sigma, n, x)\}_{n=1}^{\infty}$ is a Cauchy sequence: by Lemma 7.18

$$d(\phi(\sigma, m, x), \phi(\sigma, n, x)) \leq Ds^{\min\{m,n\}} \text{ for all } x \in K,$$

and the right-hand side here tends to zero as m and n tend to infinity. The uniformity of the convergence follows from the fact that the constant K is independent of $x \in K$.

Next we prove that $\phi : \Sigma_N \to A$ is continuous. Let $\epsilon > 0$. Choose n so that $s^n D < \epsilon$, and let $\sigma, \alpha \in \Sigma$ obey

$$d_c\{\sigma, \alpha\} < \sum_{m=n+2}^{\infty} \frac{N}{(N+1)^m} = \frac{1}{(N+1)^{n+1}}.$$

Then one can verify that σ must agree with α through n terms; that is, $\sigma_1 = \alpha_1$, $\sigma_2 = \alpha_2$,...,$\sigma_n = \alpha_n$. It follows that for each $m \leq n$ we can write

$$d(\phi(\sigma, m, x), \phi(\alpha, m, x)) = d(\phi(\sigma, n, x_1), \phi(\alpha, n, x_2)),$$

for some pair $x_1, x_2 \in \tilde{K}$. By Lemma 7.18 the right-hand side here is smaller than $s^n D$ which is smaller than ϵ. Taking the limit as m approaches infinity we find

$$d(\phi(\sigma), \phi(\alpha)) < \epsilon.$$

Finally, we prove that ϕ is onto. Let $a \in A$. Then, since

$$A = \lim_{n \to \infty} W^n(\{x\}),$$

it follows from Theorem 2.7 there is a sequence $\{\alpha^n \in \Sigma_N : n \in \mathbb{N}\}$ such that

$$\lim_{n \to \infty} \phi(\alpha^n, n, x) = a.$$

Since (Σ_N, d_c) is compact, it follows that $\{\alpha^n : n \in \mathbb{N}\}$ possesses a convergent subsequence with limit $\alpha \in \Sigma_N$. Without loss of generality assume $\lim_{n \to \infty} \alpha^n = \alpha$. Then the number of successive initial agreements between the components of $\alpha^{(n)}$ and α increases without limit. That is, if

$$\gamma(n) = \text{ number of elements in } \{j \in \mathbb{N} : \omega_k^{(n)} = \omega_k \text{ for } 1 \leq k \leq j\},$$

then $\gamma(n) \to \infty$ as $n \to \infty$. It follows that

$$d(\phi(\alpha, n, x), \phi(\alpha^{(n)}, n, x)) \leq s^{\gamma(n)} D.$$

By taking the limit on both sides as $n \to \infty$ we find $d(\phi(\alpha), a) = 0$, which implies $\phi(\alpha) = a$. Hence $\phi : \Sigma \to A$ is onto.[47] ▮

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[47] This theorem and proof are taken from Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993. pp. 123-5. I intend to rewrite the proof of the final draft.

8 References

Barnsley, Michael F. *Fractals Everywhere*, second edition. 1993.

Bryant, Victor. Metric spaces: iteration and application. 1985.

Holmgren, Richard A. *A first course in discrete dynamical system*, second edition. 1996.

Schumacher, Carol S. *The Analysis Tree*, Spring 1999 version.

Schumacher, Carol S. *Chapter Zero*. 1996.