Compactness, Connectedness, and Continuity: An Analysis of the Cantor No-Middle-Third Set

Joey Neilsen

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Kenyon College
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1 A Note to the Reader

In April of 2005, I was talking to Carol Schumacher about possibilities for a worthy Senior Exercise. Some time before I had heard that every compact metric space was the continuous image of some set called the Cantor Set. Carol and I had been discussing space-filling curves, near enough conceptually that the topic soon turned to the Cantor Set. Carol told me a few of the more intriguing properties of the set, showed me geometrically how to construct it, and mentioned the theorem above as a possible final proof for a Senior Exercise. “That’s a good proof. I think you might be able to do it,” she said. “You might have to look it up, but it would be worth a shot.” I was somewhat intimidated by the thought of working on the Senior Exercise without looking up any of the proofs, especially the hard ones, but I have never been one to resist a challenge.

The following work is entirely my own. I have frequently consulted *Mathworld, Planetmath,* and Carol Schumacher’s Real Analysis textbook *Closer and Closer: An Introduction to Real Analysis* for definitions and available theorems, but, with the exception of the theorems on convergence of sequences of functions, which we covered in Real Analysis II, as well as Cantor’s Diagonalization Argument, I had not seen any of the following proofs until I finished them (and I had to “finish” several of them a few times before I could actually claim to have seen the proof).

That said, I owe a great debt of gratitude to Carol for all her assistance. She has happily made herself available for questions and the occasionally necessary suggestion. She has let me present my proofs and told me, in more or less delicate terms, why they were incorrect, and has been quite helpful as I have worked on themes and the presentation of the following work, and for all of this I am most grateful.

2 Introduction

In so many Calculus classes, we think mainly about continuous functions and intervals of real numbers. We rarely deal with functions on disconnected domains, and in fact the idea of a function, much less a continuous one, on a heavily disconnected domain is entirely foreign. This is no surprise, because even in dealing with disconnected sets, we tend to think of a few large disjoint "pieces" of set.

However, it will be our task in the following pages to get acquainted with the Cantor Set, which is something like the interval from zero to one after being so completely pulverized that every one of its subsets with two or more elements is disconnected. As we shall see, it is a set of length zero with no interior, composed entirely of boundary points, but also composed entirely of limit points. Once familiar with its many properties, we will find a most interesting connection between the Cantor Set, continuous functions, and compact metric spaces.

The Cantor Set, also known as the Cantor No-Middle-Third Set, can be constructed as follows:

1. Define $T_0 = [0, 1]$.
2. Remove the middle third of T_0 as an open interval and define T_1 to be the remaining set of points, i.e. $T_1 = [0, 1] \setminus (1/3, 2/3)$.
3. Remove the middle thirds of each interval comprising T_1 and define the remaining points to be T_2.
4. Repeat recursively.
5. The Cantor Set T_∞ is the intersection of all T_n.

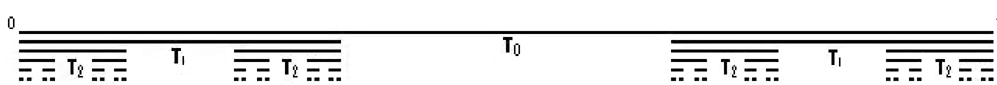

Figure 1: Stages, T_n, of the Cantor Set.

This set of instructions, in conjunction with Figure 1, gives us something of an idea of each stage, T_n, of the Cantor Set. But we would like to see a mathematical definition of each stage, and ultimately a mathematical form for the Cantor Set. We might think of the construction of this set this way: each successive step takes out every other interval (where these intervals decrease in length as 3^{-n}). If we allow this effect to accumulate, i.e. we don't 'reinsert' any points once we have removed them, we will come up with the stages T_n.

Mathematically, the n^{th} step removes from $[0,1]$ the union $\bigcup_k \left(\frac{2k+1}{3^n}, \frac{2k+2}{3^n}\right)$. But over what ks do we take the union? From our brief view of the set in Figure 1, we could guess that the largest upper endpoint $x_{\max} = (3^n - 1)/3^n$. Then $2k_{\max}+2 = 3^n - 1$, so it must be that $k_{\max} = (3^n - 3)/2 = 3(3^{n-1} - 1)/2$. For the skeptic, 3^{n-1} is odd, so $3^{n-1} - 1$ is even. Then $(3^{n-1} - 1)/2$ is an integer, and thus $k_{\max}$ is an integer. Naturally, $k_{\min} = 0$. So we know that at the n^{th} level, we subtract the set

$$\bigcup_{k=0}^{\frac{3}{2}(3^{n-1}-1)} \left(\frac{2k+1}{3^n}, \frac{2k+2}{3^n}\right)$$

from the previous stage. In order to define each stage independently of its precursors, we shall subtract out *every* necessary interval at each stage. That is, we shall subtract the set

$$\bigcup_{i=1}^{n} \bigcup_{k=0}^{\frac{3}{2}(3^{i-1}-1)} \left(\frac{2k+1}{3^i}, \frac{2k+2}{3^i}\right)$$

from the interval $[0,1]$ to define the n^{th} stage. Then we can easily define

$$T_n = [0,1] \setminus \bigcup_{i=1}^{n} \bigcup_{k=0}^{\frac{3}{2}(3^{i-1}-1)} \left(\frac{2k+1}{3^i}, \frac{2k+2}{3^i}\right).$$

Finally,

$$\begin{aligned} T_\infty &= \bigcap_{n\in\mathbb{N}} T_n \\ &= \bigcap_{n\in\mathbb{N}} \left([0,1] \setminus \bigcup_{i=1}^{n} \bigcup_{k=0}^{\frac{3}{2}(3^{i-1}-1)} \left(\frac{2k+1}{3^i}, \frac{2k+2}{3^i}\right) \right). \end{aligned}$$

Of course, we would be interested in defining the Cantor Set as the intersection of the union of some disjoint intervals, but it turns out that the approach we have used is slightly easier to deal with.

Before we continue, we should point out a few characteristics of the stages that will be important later. As is clear from Figure 1 and somewhat evident from our definition of T_n, the n^{th} stage is composed of 2^n disjoint subintervals of length 3^{-n}. Furthermore, the minimum distance between any two points in distinct subintervals is 3^{-n}. While we make these statements without proof, we may certainly point out that some consideration of the definition of T_n and a few simple induction proofs would show us to be correct on all counts.

Now that we have defined the Cantor Set, we are in a position to ask questions: Is it open, closed, or neither? Is it empty, finite, countable, or uncountable? Can we describe its elements individually? We shall begin with set theoretic properties, because describing the elements of this set will turn out to require a little work.

3 Closed and Compact

Our first claim is that T_∞ is a closed set, meaning that it contains all of its limit points. Since this and the claim that the Cantor set is compact require only very short proofs, we will list them as a single theorem.

Theorem 1: The Cantor Set T_∞ is closed and compact.

Proof:

We shall proceed directly. Fix $n \in \mathbb{N}$. We know that $[0,1]$ is closed, and we know that the interval $((2k+1)3^{-n}, (2k+2)3^{-n})$ is open for all $k \in \mathbb{Z}$. Then

$$\bigcup_{k=0}^{\frac{3}{2}(3^{n-1}-1)} \left(\frac{2k+1}{3^n}, \frac{2k+2}{3^n} \right)$$

is an open set because every union of open sets is open. Similarly,

$$\bigcup_{i=1}^{n} \bigcup_{k=0}^{\frac{3}{2}(3^{i-1}-1)} \left(\frac{2k+1}{3^i}, \frac{2k+2}{3^i} \right)$$

is open. Therefore

$$T_n = [0,1] \setminus \bigcup_{i=1}^{n} \bigcup_{k=0}^{\frac{3}{2}(3^{i-1}-1)} \left(\frac{2k+1}{3^i}, \frac{2k+2}{3^i} \right)$$

$$= \quad [0,1] \bigcap \left(\bigcup_{i=1}^{n} \bigcup_{k=0}^{\frac{3}{2}(3^{i-1}-1)} \left(\frac{2k+1}{3^i}, \frac{2k+2}{3^i} \right) \right)^{\mathcal{C}}$$

is closed, because the complement of an open set is closed, and any intersection of closed sets is closed. Finally, because

$$T_\infty = \bigcap_{n \in \mathbb{N}} T_n,$$

we can conclude that the Cantor Set is closed (again because every intersection of closed sets is closed).

As noted, we would also like to demonstrate that the Cantor Set is compact, which means that any open cover for T_∞ has a finite subcover. To prove this claim, we need only notice that the Cantor Set is bounded by 0 and 1, and recall that the closed and bounded subsets of $\mathbb{R}$ are exactly the compact subsets of $\mathbb{R}$. Thus the Cantor set is compact. ∎

So now we know that T_∞ contains all of its limit points and that it is not too 'big' or spread out. But if we take a moment to consider what we know at this point, we are sure to realize that we know very little. Is the countable or uncountable? For all we know, it could be the empty set! Is it connected or dense? To answer these questions, we need to a precise description of the elements of T_∞, and for this, we need a brief foray into base 3, or ternary, numbers.

4 Ternary Expansion

We are, of course, familiar with the concept that any integer can be written as the sum of distinct powers of two. For example, $73 = 64 + 8 + 1 = 2^6 + 2^3 + 2^0 =_2 1001001$. This last term is the binary expression for 73. But we could do the same with 3 to get a ternary expansion: $73 = 54 + 18 + 1 = 2 \cdot 3^3 + 2 \cdot 3^2 + 1 \cdot 3^0 =_3 2201$. But 73 is larger than 1, and we would like to know if we can represent any real number in the interval $[0, 1]$ in ternary form (we will need to use negative powers of 3, naturally). At this point the reader may wonder why we are so interested in threes, but a quick glance at Figure 1 may provide sufficient motivation - the Cantor Set divides this interval into thirds, ninths, twenty-sevenths, and so on.

Our claim, then, is that $\forall\ x \in [0,1],\ x$ can be written in base 3. We shall restate this claim for a reason that will become clear shortly:
Theorem 2: $\forall\ x \in [0,1],\ \exists\ (a_n)$ such that for all $n,\ a_n \in \{0,1,2\}$, and

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}.$$

Proof:

Case 1: $x = 1$. We shall proceed directly. For all $n \in \mathbb{N}$, let $a_n = 2$. Then

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{a_i}{3^i} &= \sum_{i=1}^{\infty} \frac{2}{3^i} \\
&= 2\sum_{i=1}^{\infty} \frac{1}{3^i} \\
&= 2\frac{1/3}{1-1/3} \\
&= 2\frac{1/3}{2/3} \\
&= 1 \\
&= x.
\end{aligned}$$

Case 2: $x < 1$. We shall proceed with an induction proof, i.e. we will show that for all $n \in \mathbb{N}$, there exist $a_1, a_2, \ldots, a_n \in \{0,1,2\}$ such that

$$0 < x - \sum_{i=1}^{n} \frac{a_i}{3^i} < \frac{1}{3^n}.$$

Base Case: $n = 1$. Choose $a_1 \in \{0,1,2\}$ such that $0 < x - \frac{a_1}{3} < \frac{1}{3}$. If we consider $[0, \frac{1}{3}),\ [\frac{1}{3}, \frac{2}{3})$, and $[\frac{2}{3}, 1)$, it will be obvious that we can do this. Then

$$0 < x - \sum_{i=1}^{1} \frac{a_i}{3^n} < \frac{1}{3}.$$

Inductive Step: Fix $n \in \mathbb{N}$ and suppose that there exist $a_1,\ a_2, \ldots,\ a_n \in \{0,1,2\}$ such that

$$0 < x - \sum_{i=1}^{n} \frac{a_i}{3^i} < \frac{1}{3^n}.$$

Then it must be that

$$x - \sum_{i=1}^{n} \frac{a_i}{3^i} \in \left[0, \frac{1}{3^{n+1}}\right),$$

$$x - \sum_{i=1}^{n} \frac{a_i}{3^i} \in \left[\frac{1}{3^{n+1}}, \frac{2}{3^{n+1}}\right),$$

or

$$x - \sum_{i=1}^{n} \frac{a_i}{3^i} \in \left[\frac{2}{3^{n+1}}, \frac{3}{3^{n+1}}\right).$$

So, by the same reasoning we used in the base case, we can conclude that there must be an $a_{n+1} \in \{0, 1, 2\}$ such that

$$0 < x - \sum_{i=1}^{n} \frac{a_i}{3^i} - \frac{a_{n+1}}{3^{n+1}} < \frac{1}{3^{n+1}}.$$

That is,

$$0 < x - \sum_{i=1}^{n+1} \frac{a_i}{3^i} < \frac{1}{3^{n+1}}.$$

So for all $n \in \mathbb{N}$, there exist $a_1,\ a_2, \ldots,\ a_n \in \{0, 1, 2\}$ such that

$$0 < x - \sum_{i=1}^{n} \frac{a_i}{3^i} < \frac{1}{3^n}.$$

Then we can choose a sequence (a_n) composed of zeros, ones, and twos such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}.$$

$\blacksquare$

5 Elements of T_∞

Now we know that we can write any number in the interval from zero to one as some weighted sum of powers of three (or one-third, to be a little clearer). It turns out (this is a strange discovery indeed) that the elements of the Cantor Set are precisely those numbers whose ternary expansion sequence (a_n)

includes absolutely no 1s. We will see this algebraically as well as graphically, as we draw comparisons between the Cantor Set and the binary tree.

First, however, we need to point out a few things about these sums of powers of three at which we will be looking. We have said that any expansion that includes a coefficient of 1 will not be included in the Cantor Set, but the reader may have realized that there are multiple ways to express sums such as these, so we must clarify with a lemma about geometric series. Fix $N \in \mathbb{N}$.

$$\begin{aligned} \sum_{i=N}^{\infty} \frac{2}{3^i} &= 2\sum_{i=N}^{\infty} \frac{1}{3^i} \\ &= 2\frac{1/3^N}{1-1/3} \\ &= 2\frac{1/3^N}{2/3} \\ &= \frac{1}{3^{N-1}}. \end{aligned}$$

Then if we see a sequence $(a_n) = (a_1, a_2, \ldots, a_{N-2}, 1, 0, 0, 0, \ldots)$, we may take it to be the sequence $(a_n^*) = (a_1, a_2, \ldots, a_{N-2}, 0, 2, 2, 2, \ldots)$. Similarly, we will rewrite the sequence $(b_n) = (b_1, b_2, \ldots, b_{N-2}, 1, 2, 2, 2, \ldots)$ as $(b_n^*) = (b_1, b_2, \ldots, b_{N-2}, 2, 0, 0, 0, \ldots)$. We will never remove more than a single 1 via this process (it will be the last 1 in a sequence if such a thing exists), and we will never use it to add a 1 to the coefficient sequence. With this in mind, we are ready for the proof.

Theorem 3: Let $x \in (0, 1)$ and let (a_n) be the sequence in $\{0, 1, 2\}$ corresponding to the ternary expansion of x. Suppose that $N \in \mathbb{N}$ is the smallest integer for which $a_N = 1$, and suppose that it is not true that $a_i = 0 \ \forall \ i > N$ or that $a_i = 2 \ \forall \ i > N$[1]. Then $x \notin T_\infty$. If (a_n) is entirely 0s and 2s, then $x \in T_\infty$.

Proof:
We shall proceed directly. We know that

$$0 < \sum_{i=1}^{N-1} \frac{a_i}{3^i} + \frac{a_N}{3^N}$$

[1]Given the preceding paragraph, it should make sense why we require this condition.

$$
\begin{aligned}
&< \sum_{i=1}^{\infty} \frac{a_i}{3^i} \\
&< \sum_{i=1}^{N-1} \frac{a_i}{3^i} + \frac{a_N}{3^N} + \sum_{i=N+1}^{\infty} \frac{2}{3^i} \\
&< 1.
\end{aligned}
$$

But then

$$\sum_{i=1}^{N-1} \frac{a_i}{3^i} + \frac{a_N}{3^N} < x < \sum_{i=1}^{N-1} \frac{a_i}{3^i} + \frac{a_N}{3^N} + \sum_{i=N+1}^{\infty} \frac{2}{3^i}.$$

But we also know that

$$
\begin{aligned}
\sum_{i=1}^{N-1} \frac{a_i}{3^i} &= \frac{a_1}{3} + \frac{a_2}{9} + \ldots + \frac{a_{N-1}}{3^{N-1}} \\
&= \frac{1}{3^N}(3^{N-1}a_1 + 3^N a_2 + \ldots + 3a_{N-1}).
\end{aligned}
$$

But a_i is even (either 0 or 2) for all $i < N$, so $3^{N-i}a_i$ is even for all $i < N$. Thus there exists $k \in \mathbb{N}$ such that

$$\sum_{i=1}^{N-1} \frac{a_i}{3^i} = \frac{2k}{3^N}.$$

Now, returning to our above inequalities and substituting, we have

$$
\begin{aligned}
\frac{2k}{3^N} + \frac{a_N}{3^N} \;&<\; x \;< \frac{2k}{3^N} + \frac{a_N}{3^N} + \sum_{i=N+1}^{\infty} \frac{2}{3^i} \\
\frac{2k}{3^N} + \frac{1}{3^N} \;&<\; x \;< \frac{2k}{3^N} + \frac{1}{3^N} + \sum_{i=N+1}^{\infty} \frac{2}{3^i} \\
\frac{2k+1}{3^N} \;&<\; x \;< \frac{2k+1}{3^N} + \frac{1}{3^N} \\
\frac{2k+1}{3^N} \;&<\; x \;< \frac{2k+2}{3^N}.
\end{aligned}
$$

But we specified earlier that the right hand side of this inequality must be less than 1, so $(2k+2)3^{-N} < 1 \Rightarrow 2k+2 \leq 3^N - 1$. Then we can conclude, surprisingly, that

$$k \leq \frac{3}{2}(3^{N-1} - 1).$$

So $x \in ((2k+1)3^{-N}, (2k+2)3^{-N})$ for some $k \leq 3(3^{N-1}-1)/2$, and therefore

$$x \in \bigcup_{i \leq N} \bigcup_{k=0}^{\frac{3}{2}(3^{i-1}-1)} \left(\frac{2k+1}{3^i}, \frac{2k+2}{3^i} \right).$$

Then $x \notin T_N$, so $x \notin T_\infty$!

Claim 2: If (a_n) has no 1s, then $x \in T_\infty$. This argument is quite similar to the one above. Fix $N \in \mathbb{N}$. Recall from above that

$$\sum_{i=N+1}^{\infty} \frac{2}{3^i} = \frac{1}{3^N}.$$

Also, notice that we could easily show from the above derivation that there must exist a $j \in \mathbb{N}$ such that

$$\sum_{i=1}^{N} \frac{a_i}{3^i} = \frac{2j}{3^N}.$$

Then

$$\begin{aligned}
\sum_{i=1}^{N} \frac{a_i}{3^i} &= \frac{2j}{3^N} \\
&\leq x \\
&= \sum_{i=1}^{N} \frac{a_i}{3^i} + \sum_{i=N+1}^{\infty} \frac{a_i}{3^i} \\
&\leq \frac{2j}{3^N} + \sum_{i=N+1}^{\infty} \frac{2}{3^i} \\
&= \frac{2j}{3^N} + \frac{1}{3^N} \\
&= \frac{2j+1}{3^N}.
\end{aligned}$$

Notice that $2j$ is even and $2j+1$ is odd. But then it cannot be that there exists a $k \in \mathbb{N}$ such that

$$x \in \left(\frac{2k+1}{3^N}, \frac{2k+2}{3^N} \right),$$

because $2k+1$ is odd and $2k+2$ is even. Therefore

$$x \notin \bigcup_{i=1}^{N} \bigcup_{k=0}^{\frac{3}{2}(3^{i-1}-1)} \left(\frac{2k+1}{3^i}, \frac{2k+2}{3^i} \right).$$

Then, because x is in every T_N, $x \in T_\infty$. ∎

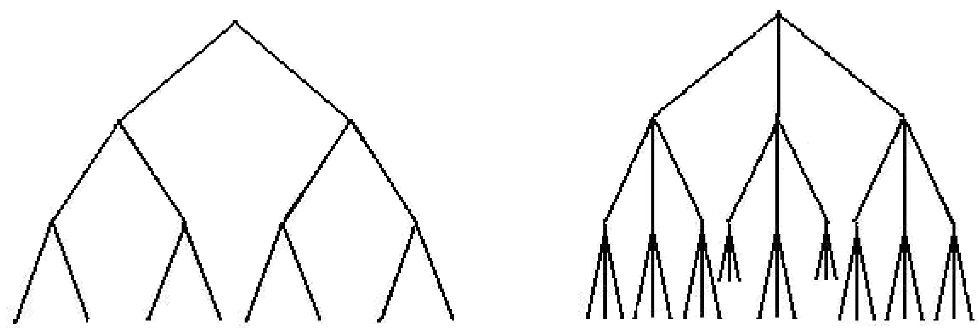

Figure 2: The binary and ternary trees.

This certainly seems to be a strange result. Why is it that any number with a 1 in its ternary expansion is not in the Cantor Set? It is not obvious from our definition of the set, and certainly not from Figure 1. But let's consider Figure 1 more carefully. We see that each stage divides every subinterval of the previous stage into three subintervals (one of which is excluded from T_∞). We can represent this process as a "ternary" tree (Figure 2). We show the binary tree, which is more familiar, for reference.

The relationship between the ternary tree and the Cantor stages is even more obvious if we overlay Figures 1 and 2 (Figure 3). We can see clearly that if we take the middle branch at any junction of the ternary tree, we drop out of the Cantor Set immediately. This leaves us with a binary tree which, it seems, may represent elements of T_∞. Now our task becomes to find the connection between 1s in ternary expansions and the middle branches of the ternary tree. Fortunately, this will not present much of a challenge.

Suppose we are given a sequence (a_n) of ternary expansion coefficients. Our goal is to translate this sequence into a set of 'directions' along the ternary tree. Recall that earlier we divided the interval [0,1] into three subintervals: $\left[0, \frac{1}{3}\right)$, $\left[\frac{1}{3}, \frac{2}{3}\right)$, and $\left[\frac{2}{3}, \frac{3}{3}\right)$. This gives us a natural choice of directions: if $a_1 = 0$, go to the subinterval $\left[0, \frac{1}{3}\right)$; if $a_1 = 1$, go to the subinterval $\left[\frac{1}{3}, \frac{2}{3}\right)$; finally, if $a_1 = 2$, go to the subinterval $\left[\frac{2}{3}, \frac{3}{3}\right)$.

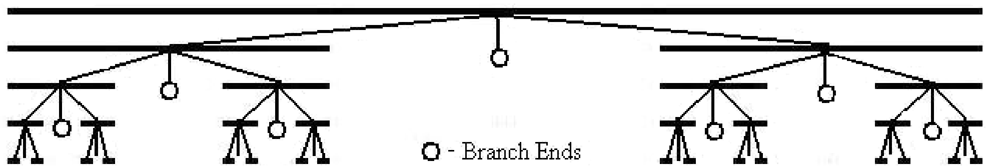

Figure 3: The Cantor Stages with the ternary tree. Notice that elements of [0,1] following middle branches will never be included in the Cantor Set.

We can clearly generalize to say that if $a_1 = 0$, we go to the left subinterval (take the left branch of the ternary tree), if $a_1 = 1$, we go to the middle subinterval (take the middle branch of the ternary tree), and if $a_1 = 2$, we go to the right subinterval (take the right branch of the ternary tree)[2]. But we already know what happens if we take any middle branch - we are excluded from the Cantor Set. Then, since we have drawn a connection between 1s in the ternary expansion sequence and these middle branches of the ternary tree, it makes sense in retrospect that sequences including 1s should not correspond to elements of T_∞.

For the sake of clarity, let's summarize this argument. We showed that there is a one-to-one correspondence between elements of the interval $[0, 1]$ and sequences in $\{0, 1, 2\}$ (by way of ternary expansions). We also showed that, in a natural way, 1s correspond to middle branches of the ternary tree. Finally, we demonstrated that middle branches of the ternary tree correspond to subintervals of [0,1] which are excluded from the Cantor Set. Thus we can be much more comfortable with our conclusion from above. If $x \in T_\infty$, then the ternary expansion sequence (a_n) corresponding to x has absolutely no 1s.

But now we are left with a one-to-one correspondence between elements of the Cantor Set and branches of the binary tree (another reason we showed the binary tree in Figure 2). As any second-year math student should know, the binary tree has uncountably many branches. We can prove this via Cantor's Diagonalization Argument, and we will do so for reference, as one cannot see this argument too many times. Instead of the tree, we will deal with sequences of 0s and 2s because they remind us conveniently of Cantor Set

[2]If the reader is interested, it would be possible to prove, based on this set of directions, that if x is an endpoint of T_n for some n and (a_n) is the ternary expansion sequence for x, then there exists $N \in \mathbb{N}$ such that $\forall\ i > N\ a_i = 0$ or $\forall\ i > N,\ a_i = 2$.

elements and because they are easier to write than "follow the left branch."

Theorem 4: The set of sequences of 0s and 2s ($S\{0,2\}$) is uncountable.

Proof:
Define a function $f : \mathbb{N} \to S\{0,2\}$ by

$$f(n) = (s_{1n}, s_{2n}, s_{3n}, \ldots),$$

where s_{jn} is the n^{th} term of some sequence s_j. Now choose $b = (b_1, b_2, b_3, \ldots)$ such that for all $i \in \mathbb{N}$,

$$b_i = \begin{cases} 0 & \text{if } s_{ii} = 2 \\ 2 & \text{if } s_{ii} = 0. \end{cases}$$

Now notice that for all $n \in \mathbb{N}$, $f(n) \neq b$, for by construction, $(f(1))_1 \neq b_1$, $(f(2))_2 \neq b_2$, and so on. Since f is arbitrary, there can be no function from the natural numbers onto the set of sequences of zeros and twos (or the Cantor Set). Then both the Cantor Set and the set of sequences of zeros and twos must be uncountably infinite! ∎

Now we are justified if we are a little confused. Looking at Figure 1, we are sure to notice that the stages seem to decimate the interval [0,1], and that what is left of this interval in the Cantor Set appears to be no more than dust. But this is not the case; in fact, T_∞ is exactly the same size as [0,1]! But our grasp of these uncountably many Cantor elements is still tenuous. Yes, we know what their ternary expansions look like, but is that the best we can do? To be honest, it is pretty close.

A final glance at Figure 1 would suggest that the endpoints of every subinterval of every stage are contained in T_∞. The proof is not very difficult, so we shall show it here.

Theorem 5: Fix $N \in \mathbb{N}$. Then the endpoints of T_N are elements of T_∞.

Proof:

Part 1. Let $x \in T_{N+1}$. Then by definition of T_{N+1},

$$x \notin \bigcup_{i \leq N+1} \bigcup_{k=0}^{\frac{3}{2}(3^{i-1}-1)} \left(\frac{2k+1}{3^i}, \frac{2k+2}{3^i} \right),$$

so

$$x \notin \bigcup_{i \leq N} \bigcup_{k=0}^{\frac{3}{2}(3^{i-1}-1)} \left(\frac{2k+1}{3^i}, \frac{2k+2}{3^i} \right).$$

Then $x \in T_N$.

Part 2. Choose $k \leq 3(3^{N-1} - 1)/2$. Then $x_1 = (2k+1)/3^N$ and $x_2 = (2k+2)/3^N$ are endpoints of subintervals of T_N. Notice that

$$\begin{aligned} x_1 &= \frac{2k+1}{3^N} \\ &= \frac{3(2k+1)}{3^{N+1}} \\ &= \frac{6k+3}{3^{N+1}} \\ &= \frac{2(3k+1)+1}{3^{N+1}} \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{2k+2}{3^N} \\ &= \frac{3(2k+2)}{3^{N+1}} \\ &= \frac{6k+6}{3^{N+1}} \\ &= \frac{2(3k+2)+2}{3^{N+1}}. \end{aligned}$$

This should remind us of the endpoints of intervals in the definition of the stages T_n. Now notice that

$$\begin{aligned} 2(3k+2)+2 &= 3(2k+2) \\ &\leq 3(3^N - 1) \\ &< 3^{N+1} - 1 \\ 2(3k+2) &\leq 3^{N+1} - 3 \\ (3k+2) &\leq \frac{3}{2}(3^N - 1). \end{aligned}$$

That is, x_2 is an endpoint of T_{N+1}! A similar argument will hold for x_1. So every endpoint is an endpoint of every successive stage. And because every element of T_N is in every previous stage, we can conclude that every endpoint is in every stage. Thus every endpoint is in the Cantor Set. ∎

Unfortunately, that is just about the best we can do. We can make up a few other sequences, e.g. alternating zeros and twos, but we must be satisfied knowing that all endpoints are in T_∞, and that, as there are only countably many endpoints, there are uncountably many Cantor Set elements that we can never find.

6 Fortunately, the Cantor Set is Perfect

At this point, the reader might be somewhat discouraged. It seems that in order to get a handle on T_∞, we need to find uncountably many sequences of zeros and twos, a task we are not prepared to take on. But we are comfortable with series and sequences in general, and the relevant powers-of-three series are particularly easy to comprehend; perhaps we can use this fact to our advantage. We will show that the Cantor Set is perfect: every element of T_∞ is a limit point of T_∞, so we can find a sequence in T_∞ approaching any Cantor Set element.

Theorem 6: The Cantor Set is a perfect set.

Proof:
Let $x \in T_\infty$. Then there exists (a_i) in $\{0, 2\}$ such that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}.$$

Now fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $2/3^N < \epsilon$. Define (b_n) to be a sequence of sequences in $\{0, 2\}$ where $b_n = b_{n_1}, b_{n_2}, b_{n_2}, \ldots$ with the following property:

$$\forall\ i \in \mathbb{N},\ b_{i_i} = \begin{cases} 0 & \text{if } a_i = 2 \\ 2 & \text{if } a_i = 0. \end{cases}$$

and $b_{i_j} = a_j$ if $j \neq i$. That is, the j^{th} sequence of (b_n) differs from (a_n) only in their j^{th} terms. This results in sequences:

$$\begin{array}{rcl} (a_n) & = & a_1, a_2, a_3, a_4, \ldots \\ b_1 & = & \mathbf{b}_{1_1}, a_2, a_3, a_4, \ldots \\ b_2 & = & a_1, \mathbf{b}_{2_2}, a_3, a_4, \ldots \\ b_3 & = & a_1, a_2, \mathbf{b}_{3_3}, a_4, \ldots \end{array}$$

and so on. Now fix $n > N$. Then

$$\begin{aligned}
\left| x - \sum_{i=1}^{\infty} \frac{b_{n_i}}{3^i} \right| &= \left| \sum_{i=1}^{\infty} \frac{a_i}{3^i} - \sum_{i=1}^{\infty} \frac{b_{n_i}}{3^i} \right| \\
&= \left| \sum_{i=1}^{n-1} \frac{a_i}{3^i} + \frac{a_n}{3^n} + \sum_{i=n+1}^{\infty} \frac{a_i}{3^i} - \sum_{i=1}^{n-1} \frac{b_{n_i}}{3^i} - \frac{b_{n_n}}{3^n} - \sum_{i=n+1}^{\infty} \frac{b_{n_i}}{3^i} \right| \\
&= \left| \sum_{i=1}^{n-1} \frac{a_i}{3^i} + \frac{a_n}{3^n} + \sum_{i=n+1}^{\infty} \frac{a_i}{3^i} - \sum_{i=1}^{n-1} \frac{a_i}{3^i} - \frac{b_{n_n}}{3^n} - \sum_{i=n+1}^{\infty} \frac{a_i}{3^i} \right| \\
&= \left| \frac{a_n}{3^n} - \frac{b_{n_n}}{3^n} \right| \\
&= \frac{2}{3^n} \\
&< \frac{2}{3^N} \\
&< \epsilon.
\end{aligned}$$

∎

So we can get arbitrarily close to $x \in T_\infty$ with other elements of T_∞. So we might feel a little better now, because this previous theorem has given us a hint that Cantor Set elements are close together in some sense, and we are always happy with converging sequences. But we should not be too comfortable, and to see why, we will conclude our investigation of the set and element properties of the Cantor Set with three rather unsettling theorems, all of which we will find to be conceptually at odds with the theorem we have just proved.

7 T_∞ is Totally Disconnected

What about subsets of the Cantor Set? Are there any conclusions we can draw about a typical subset of T_∞? Consider some $\Gamma \subseteq T_\infty$. We know that the Cantor Set is full of holes, which are distributed all throughout the interval [0,1] and come in many sizes. We could be sure, then, that at some point there will be a hole between one element of Γ and another element of Γ. If that is the case, we should be able to write Γ as the disjoint union of two open sets. Equivalently, we could conclude that Γ is disconnected. Since

Γ was arbitrary, we say that T_∞ is *totally disconnected.*

Definition 1: A set Λ is *totally disconnected* if every subset of Λ with more than one element is disconnected[3].

Theorem 7: T_∞ is totally disconnected.

Proof:
Let Γ be some subset of the Cantor Set with more than one element. Since Γ is a nonempty subset of $\mathbb{R}$ which is bounded above and below, we are guaranteed the existence of a supremum and infimum. Then let $s = \sup(\Gamma)$ and $i = \inf(\Gamma)$. Choose $\epsilon < \min(i, s-i)$, and $N \in \mathbb{N}$ such that $3^{-N} < \epsilon/2$. Fix $k \in \mathbb{Z}$ such that

$$\frac{2k}{3^N} \leq i < \frac{2k+1}{3^N}.$$

The reader will not find it difficult to show that $0 \leq k \leq 3(3^{N-1}-1)/2$, an inequality which might suggest a subinterval of [0,1] excluded from T_∞ if we recall the definition of that set. Now notice that

$$\begin{aligned}
\frac{2k}{3^N} &\leq i \\
&< \frac{2k+1}{3^N} \\
&< \frac{2k+2}{3^N} \\
&< \frac{2k}{3^N} + \epsilon \\
&< i + \epsilon \\
&< s.
\end{aligned}$$

That is,

$$\left[\frac{2k+1}{3^N}, \frac{2k+2}{3^N}\right] \subset (i, s).$$

So this interval cuts through Γ, a fact we can use to our advantage (Figure 4). We will consider $A = \Gamma \cap [0, \frac{2k+1}{3^N}]$ and $B = \Gamma \cap [\frac{2k+2}{3^N}, 1]$. These are two nonempty disjoint sets whose union is equal to Γ. Furthermore, these sets are both open (in Γ). If the reader is not willing to accept this without proof, we may show it briefly. Fix $x \in A$, and let $r = \epsilon/2$ (same ϵ as above). Then

[3] Connectedness would of course have no meaning for an empty set or a singleton set.

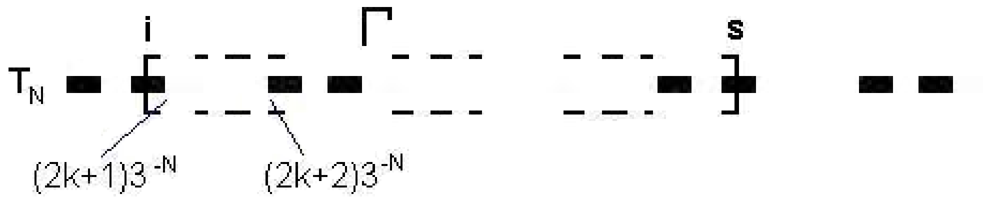

Figure 4: Some Γ shown with T_N. Notice that there are many available holes in the Cantor Set which could divide Γ.

$x + r < (2k+2)3^{-N}$, so if $b \in (x-r, x+r) \cap \Gamma$, then $b \notin B$. Then it must be that $b \in A$. So there exist nonempty disjoint open subsets A and B of Γ such that $\Gamma = A \cup B$. Equivalently, Γ is disconnected. Therefore T_∞ is totally disconnected. ∎

This is quite strange. We are used to dealing with connected sets, like intervals and other 'smooth' subsets of $\mathbb{R}$. It is likely a very small minority of the population that, in visualizing some arbitrary metric space, thinks of a disconnected set. This is not to say that we are entirely unfamiliar with discrete mathematics, but to point out that most of our experience is with connected sets or maybe a collection of intervals. But we might say that the Cantor Set has no subintervals. There is nothing 'intervaly' about T_∞; we may find it useful to think of the Cantor Set in terms of the subintervals provided by T_n, but the fact remains that any subset we choose can be chopped into infinitely many distinct and separated pieces.

8 T_∞ is Nowhere Dense

We can go farther still, though. Every element of the Cantor Set is a limit point of the Cantor Set. That is, every open ball around an element of T_∞ contains infinitely elements of T_∞. So we might be inclined to think of this set as tiny nuggets of Cantor Set spread across the interval from zero to one. Unfortunately, we would be somewhat in error to hold this view, because the interior of the Cantor Set is empty.

Lemma 1: If $x \in T_\infty$, then x is a limit point of $[0,1]\backslash T_\infty$.

Proof:
Fix $\epsilon > 0$ and $x \in T_\infty$. Choose $N \in \mathbb{N}$ such that $3^{-N} < \epsilon$. Naturally, x is in some subinterval s_j of T_N. But also note that T_{N+1} removes the middle third of s_j. Then $B_\epsilon(x)$ contains an interval in, and thus infinitely elements of, $[0,1] \setminus T_\infty$. Then x is a limit point of $[0,1] \setminus T_\infty$. ∎

Definition 2: A set Λ is *nowhere dense* if its interior, $Int(\Lambda)$, is empty.

Theorem 8: The Cantor Set is nowhere dense.

Proof:
We shall proceed directly. Recalling that T_∞ is closed, we know that $T_\infty = \overline{T_\infty}$, the closure of the Cantor Set. Now

$$Int(T_\infty) = \left(\overline{{T_\infty}^{\mathcal{C}}}\right)^{\mathcal{C}},$$

and we note that we are in the metric space [0,1]. Then ${T_\infty}^{\mathcal{C}} = [0,1] \setminus T_\infty$, and thus $\overline{{T_\infty}^{\mathcal{C}}} = ([0,1] \setminus T_\infty) \cup \{x \in [0,1] : \ x \text{ is a limit point of } [0,1] \setminus T_\infty\}$. By our lemma above, we know that $T_\infty \subseteq \{x \in [0,1] : \ x \text{ is a limit point of } [0,1] \setminus T_\infty\} \subseteq [0,1]$. Thus

$$\begin{aligned} \overline{{T_\infty}^{\mathcal{C}}} &= ([0,1] \setminus T_\infty) \cup T_\infty \\ &= [0,1]. \end{aligned}$$

That is, the closure of the complement of the Cantor Set is the interval from zero to one. We come to the conclusion that

$$\begin{aligned} Int(T_\infty) &= \left(\overline{{T_\infty}^{\mathcal{C}}}\right)^{\mathcal{C}} \\ &= [0,1]^{\mathcal{C}} \\ &= \emptyset. \end{aligned}$$

∎

So the interior of the Cantor Set is empty, despite the fact that we can find elements of T_∞ which are as close together as we like. This set is completely composed of boundary points. Though we might accept each of these proofs individually, it is difficult to combine the concepts of a perfect set and a nowhere dense set into a sensible image.

9 The Cantor Set has Lebesgue Measure Zero

So the Cantor Set has an empty interior and has only boundary points, and each of its subsets with more than one element is disconnected. We might have liked to describe the 'length' of this set, as a last effort to understand its relation to the interval [0,1]. It would seem, based on the above conclusions, that this task is beyond our grasp. But let us stop a minute and consider how we would define the length of a disjoint union of subintervals. Suppose we had the set $[0,1] \cup [2,3]$. Technically, this set does not have a length. But we would surely believe that it is as 'long' as [0,2]. It is possible to demonstrate this, but it requires that we generalize our concept of length. To do so, we introduce *Lebesgue measure*. It would be beyond the scope of this paper to give a full introduction to measure and measure theory; for this reason, we shall deal with Lebesgue measure operationally.

Definition 3: Let Θ be any measurable subset of $\mathbb{R}$ and $C = \{A_\alpha : \alpha \in \Lambda\}$, where for all $\alpha \in \Lambda$, $A_\alpha = \{A_i\}_{i=0}^{\infty}$ is any countable collection of open intervals which covers Θ. Then, denoting the length of an interval A_i as $L(A_i)$, the Lebesgue measure of Θ is

$$\mu_L(\Theta) = \inf\left(\left\{\sum_{A_i \in A} L(A_i) : A \in C\right\}\right).$$

Since this definition requires a number of other definitions, it will be instructive to go through it point by point. So Θ is any measurable subset of $\mathbb{R}$ (measurable meaning that we can assign to it a non-negative real number which is a generalization of length) and $A = \{A_i\}_{i=0}^{\infty}$ is any countable collection of open intervals which covers Θ. We define $C = \{A_\alpha : \alpha \in \Lambda\}$ to be the collection of all possible collections A. That is, every element of C is a countable collection of open intervals in $\mathbb{R}$ which covers Θ. To distinguish between length of intervals and the measure of sets, we shall denote the length of any interval I as $L(I)$. Then we shall define

$$M = \left\{\sum_{A_i \in A} L(A_i) : A \in C\right\}.$$

Strictly speaking, M is the set of summed lengths of intervals in the A collections. Colloquially, we could think of $\sum L(A_i)$ as the "length" of collection

A, in which case M is the set of lengths of elements of C. Because $L(A_i) \geq 0$ for all i and A, we are guaranteed that M is bounded below. Finally, we can define the Lebesgue measure μ_L :

$$\mu_L(\Theta) = \inf(M).$$

This is the Lebesgue measure of Θ. If Θ is an interval, then $\mu_L(\Theta) = L(\Theta)$. That is, μ_L behaves like length for intervals. It is possible to show that any closed set is measurable, but in the interest of space, we will assume this fact. So we would like to prove that the Lebesgue measure of T_∞ is equal to zero. We know $\mu_L(T_\infty)$ exists because the Cantor Set is closed, and thus measurable. Let us define the sets above with $\Theta = T_\infty$. It becomes our task to show that $\inf(M) = 0$ or, equivalently, that there exists a sequence in M which converges to zero.

Theorem 9: The Cantor Set has $\mu_L(T_\infty) = 0$.

Proof:
Recall that for each $n \in \mathbb{N}$, we can divide T_∞ into 2^n subsections of length 3^{-n}. Then of course, we could cover each of these subsections with 2^n open intervals of length $3^{-n} + 2 \cdot 3^{-(n+1)} = 5 \cdot 3^{-(n+1)}$. For example, we would cover the subsection

$$\left[\frac{2}{9}, \frac{3}{9}\right] \cap T_\infty \text{ with the open interval } \left(\frac{2}{9} - \frac{1}{27}, \frac{3}{9} + \frac{1}{27}\right) = \left(\frac{5}{27}, \frac{10}{27}\right).$$

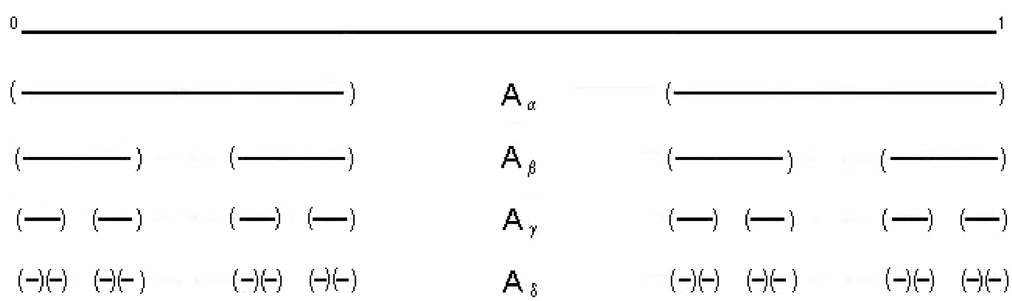

Figure 5: A few A collections which cover T_∞ at different stages. Notice how rapidly their summed lengths go to zero.

Then at most, we need 2^n intervals of length $5 \cdot 3^{-(n+1)}$: a total length of $5 \cdot 2^n \cdot 3^{-(n+1)}$. In Figure 5 we see a few of these collections of intervals; we use more smaller intervals to cover each successive stage of the Cantor Set. We know that for all $n \in \mathbb{N}$ there must exist a collection $\Delta_n \in C$ which consists of only the 2^n subintervals $(\Delta_{n_i},\ i \leq 2^n)$ which conveniently cover T_n (since C consists of all possible covering collections). Now we need a sequence of the summed lengths of these Δ_n collections, so let

$$M_n = \sum_{i=1}^{2^n} L(\Delta_{n_i}) = 5\frac{2^n}{3^{n+1}}.$$

Then

$$\begin{aligned} \lim_{n\to\infty}(M_n) &= \lim_{n\to\infty} 5\frac{2^n}{3^{n+1}} \\ &= \frac{5}{3}\lim_{n\to\infty}\left(\frac{2}{3}\right)^n \\ &= 0. \end{aligned}$$

So there is a sequence in M which converges to zero. We can see this pretty well in Figure 5: because the interval [0,1] gets chopped up into tinier and tinier pieces, we can use smaller and smaller intervals to cover T_n. Fortunately, the intervals decrease in size faster than they increase in number. Then the total length of Δ_n must converge to zero. Finally,

$$\begin{aligned} \mu_L(T_\infty) &= \inf(M) \\ &= 0. \end{aligned}$$

The Cantor Set has Lebesgue Measure 0. ∎

10 Compact Metric Spaces and the Cantor Set

As we noted above, much of undergraduate analysis seems to concern continuous functions and connected metric spaces. Continuity is a useful and sometimes powerful tool, for it gives us integrability and is a direct consequence of differentiability. In Real Analysis, we add compactness to the list of powerful mathematical tools. Compactness turns continuous functions into uniformly continuous functions and, via the completeness of compact metric spaces, gives us the limits of Cauchy sequences.

The Cantor Set shows up in interesting and sometimes unexpected ways regarding these mathematical tools, i.e. it is compact and perfect, but also totally disconnected. But what about continuity? Specifically, is there any way that we can use the many properties of the Cantor Set to find some relationship between continuous functions and compact metric spaces? The frequent reoccurrence and juxtaposition of all these properties might suggest that such a relationship exists.

We will attempt to show the most counter-intuitive property of the Cantor Set: Given an arbitrary compact metric space $\mathcal{Y}$, there exists a function $f : T_\infty \to \mathcal{Y}$ such that f is continuous and onto. This is a strange result indeed. We often take abstract spaces and define functions *on* them, the distance function being the easiest example. But these are functions from abstract spaces to the real numbers. The reader might point out here that we also have experience dealing with functions between two arbitrary metric spaces, and of course that is the case. However, this proof will require us to *construct* a continuous function f whose range is equal to its codomain, and it is the attempt to define a function from a well-defined subset of $\mathbb{R}$ onto a metric space about which we know very, very little, which we shall find to be no conceptually easy task. But first, we need to remind the reader of a few ideas about sequences of functions.

10.1 Continuity and Convergence of Functions

At this point, the reader should definitely be familiar with sequences of real numbers, or at least the few we have discussed here. But it is possible to define rather intuitively a sequence of functions on the real numbers and, consequently, to ask what it would mean for such a sequence to converge. If this sequence of functions does converge, we certainly expect it to converge to some other real-valued function, but can we determine whether or not the limit function is continuous?

First, let's try an example. We can define a pretty simple sequence of functions (f_n) as follows:

$$\forall\ n \in \mathbb{N} \text{ and } x \in \mathbb{R},\ \text{define } f_n : \mathbb{R} \to \mathbb{R} \text{ by } f_n(x) = \frac{x}{n}.$$

The terms of this sequence are lines through the origin with slope $1/n$. What happens to these terms as n gets large? Well, if we pick any value of x, say x_0, we can easily show that $(f_n(x_0)) \to 0$. (Notice, though, that as x_0 increases, this sequence will converge more slowly). We should be able to say, though, that the sequence (f_n) converges to the constant function 0.

This kind of convergence is called *pointwise convergence*, and we shall see that it is not a sufficient condition to make the necessary conclusions about continuity. But we need a better definition of this pointwise convergence.

Definition 4: Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces; let $(f_n) : \mathcal{X} \to \mathcal{Y}$ be a sequence of functions, and fix $x \in \mathcal{X}$. Then (f_n) converges at x if the sequence $(f_n(x))$ converges, and we say that (f_n) converges pointwise if $(f_n(x))$ converges for all $x \in \mathcal{X}$.

Then, if (f_n) converges pointwise, we should be able to define some limit function. We do this as follows: let $f : \mathcal{X} \to \mathcal{Y}$ be the function

$$f(x) = \lim_{n \to \infty} f_n(x);$$

then $(f_n) \to f$.

As we noted above, however, we cannot necessarily draw any conclusions about the continuity of the limit function. In the above example, of course, the limit function is constant and thus continuous, but this will not always be the case (we shall make a number of statements here without proof; the reader will not find the proofs difficult). Consider the sequence $f_n : [0,1] \to [0,1]$ given by $f_n(x) = x^n$. A few terms of this sequence are shown in Figure 6. From this figure, we can see that our sequence of functions converges to zero for all $x < 1$, but that $f_n(1) = 1 \ \forall \ n \in \mathbb{N}$. Then the limit function is discontinuous.

So we clearly need some secondary criterion in order to conclude that we have a continuous limit function. To find this criterion, we should notice something in common between our straight-line example and the x^n example: as x increases, we need to go farther and farther out into the sequence of functions for $f_n(x)$ to approach $f(x)$. Consider the second example at $x = 0.5$ and $x = 0.99$, for this claim is not entirely obvious here. At $x = 0.5$, $f_{10}(x) \sim 0$. That is, $(f_n(0.5)) \to 0$ rather quickly. But $f_{25}(0.99) = 0.7778$, and it is quite clear from Figure 6 that $f_{25}(0.99) >> 0$. So $(f_n(0.99)) \to 0$ very slowly. We have already noted similar behavior for the straight-line example, so the common theme seems to be this non-uniform convergence. If we could find

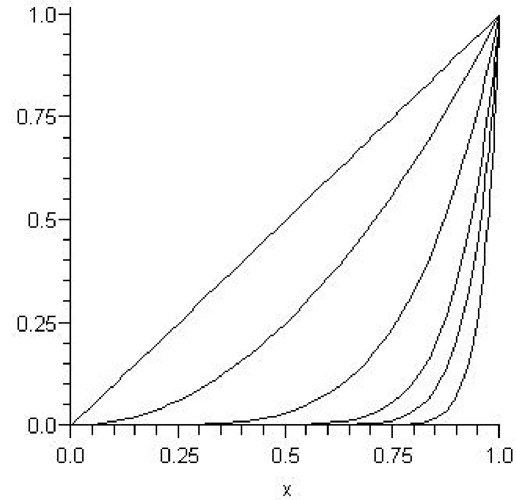

Figure 6: A series of continuous functions (x^n on [0,1]) which converge pointwise to a discontinuous function.

some function in the sequence after which all $f_n(x)$ would be "close to" $f(x)$, we might be able to conclude continuity of the limit function.

This second form of convergence is called *uniform convergence,* and we can define it more precisely as follows:

Definition 5: Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces; let $(f_n) : \mathcal{X} \to \mathcal{Y}$ be a sequence of functions. Suppose that (f_n) converges pointwise to some function $f : \mathcal{X} \to \mathcal{Y}$. Then the sequence (f_n) converges uniformly to f if $\forall\ \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n > N$ and $x \in \mathcal{X}$, then $d(f_n(x), f(x)) < \epsilon$.

Theorem 10: Suppose that (f_n) is a sequence of continuous functions which converges uniformly to f. We claim that f is continuous at x for all $x \in \mathcal{X}$.

Proof:
Fix $\epsilon > 0$. Since (f_n) converges uniformly, we may choose $N \in \mathbb{N}$ such that for all $n > N$ and all $x \in \mathcal{X}$, $d(f_n(x), f(x)) < \epsilon/3$. Fix $n > N$ and $x \in \mathcal{X}$. Because f_n is continuous at x, we may choose $\delta > 0$ such that if $d(x, y) < \delta$,

then $d(f_n(x), f_n(y)) < \epsilon/3$. Fix $y \in \mathcal{X}$ such that $d(x,y) < \delta$. Then

$$\begin{array}{rcl} d(f(x), f(y)) & \leq & d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ & < & \dfrac{\epsilon}{3} + \dfrac{\epsilon}{3} + \dfrac{\epsilon}{3} \\ & < & \epsilon. \end{array}$$

Therefore f is continuous at x. ∎

So the limit function of a uniformly convergent sequence of functions is continuous. But we will be better off still if we examine a few more points about convergent sequences.

In particular, we should discuss Cauchy sequences. The reader may recall that a sequence (a_n) is a Cauchy sequence if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $m,\ n > N$, then $d(a_n, a_m) < \epsilon$. It is rather trivial to extend this concept to a sequence of functions (using $\mathcal{X}$ and $\mathcal{Y}$ as the domain and codomain, respectively): a sequence of functions (f_n) is a *uniformly Cauchy sequence* if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n,\ m > N$ and $x \in \mathcal{X}$ then $d(f_n(x), f_m(x)) < \epsilon$.

Theorem 11: If (f_n) is a uniformly Cauchy sequence of continuous functions and $\mathcal{Y}$ is a complete metric space, then (f_n) converges uniformly to some continuous function $f : \mathcal{X} \to \mathcal{Y}$.

Proof:
Fix $\epsilon > 0$ and $x_0 \in \mathcal{X}$. Because $(f_n(x_0))$ is a Cauchy sequence and $\mathcal{Y}$ is complete, the sequence $(f_n(x_0))$ must converge. Since this conclusion doesn't depend on x_0, we find that (f_n) has a pointwise limit for each element of $\mathcal{X}$. Define $f : \mathcal{X} \to \mathcal{Y}$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Now because (f_n) is uniformly Cauchy, we may choose $N \in \mathbb{N}$ such that if $n,\ m > N$ and $x \in \mathcal{X}$, then $d(f_n(x), f_m(x)) < \epsilon/2$. Also, because $(f_i(x_0)) \to f(x_0)$, we can find $m > N$ such that $d(f_m(x_0), f(x_0)) < \epsilon/2$. Now fix $n > N$. Then

$$\begin{array}{rcl} d(f_n(x_0), f(x_0)) & \leq & d(f_n(x_0), f_m(x_0)) + d(f_m(x_0), f(x_0)) \\ & < & \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} \\ & < & \epsilon. \end{array}$$

So there exists an $N \in \mathbb{N}$ such that for all $n > N$ and $x \in \mathcal{X}$, $d(f_n(x), f(x)) < \epsilon$. Then f_n converges uniformly to f on $\mathcal{X}$. And since each term of this sequence is a continuous function, we can conclude that f is continuous. ▮

Finally we have all the necessary tools to prove our final claim, so let us delay the proof no further.

10.2 The Proof

Theorem 12: Let $\mathcal{Y}$ be any compact metric space. Then there exists a function $f : T_\infty \to \mathcal{Y}$ such that f is continuous and onto.

Proof:

Our goal here is to construct a sequence of continuous functions from T_∞ to $\mathcal{Y}$ which is uniformly convergent. We shall have to proceed recursively for a reason that will become clear shortly. We know that $\mathcal{Y}$ is a compact metric space, so any open cover for $\mathcal{Y}$ has a finite subcover; we will make plentiful use of this fact. Our plan is to use the compactness of $\mathcal{Y}$ to find for each $y \in \mathcal{Y}$ a sequence of balls of decreasing radius containing y, with each successive ball contained in the one before, and the radii of these balls converging to zero. We will find a way to map elements of the Cantor Set to these balls: our function f.

Consider the open cover $\{B_1(y)\}_{y\in\mathcal{Y}}$. Naturally, we can find a finite subset $\{B_1(y_i)\}_{i\leq N_1}$ of this collection of balls of radius 1 which covers $\mathcal{Y}$. That is, we have N_1 balls, centered at points $y_1, ..., y_{N_1}$, whose union is equal to $\mathcal{Y}$ (see Figure 7). As we are fully aware, the Cantor Set has natural collections of subintervals with sizes of all powers of two, so it would be convenient if N_1 were a power of 2. Fortunately, we can treat N_1 as desired without any loss of generality. Choose $m_1 \in \mathbb{N}$ such that $N_1 \leq 2^{m_1}$. Then we list the centers of our balls: $y_1,\ y_2,\ ...,\ y_{N_1},\ y_{N_1},\ ...,\ y_{N_1}$, repeating y_{N_1} as necessary until the list has exactly 2^{m_1} elements. Now T_{m_1} has 2^{m_1} subintervals s_j, $1 \leq j \leq 2^{m_1}$, and given any $x \in T_\infty$, there exists $j^\star \leq 2^{m_1}$ such that $x \in s_{j^\star}$. Then define $f_1 : T_\infty \to \mathcal{Y}$ as follows:

$$f(x) = y_{j^\star}.$$

To clarify, f_1 maps each of the 2^{m_1} subintervals of T_∞ to a center of one of our finite cover balls (Figure 8). We can show very easily that f_1 is continuous. Fix $\epsilon > 0$. Choose $\delta = 3^{-m_1}$. Then we know from above that if

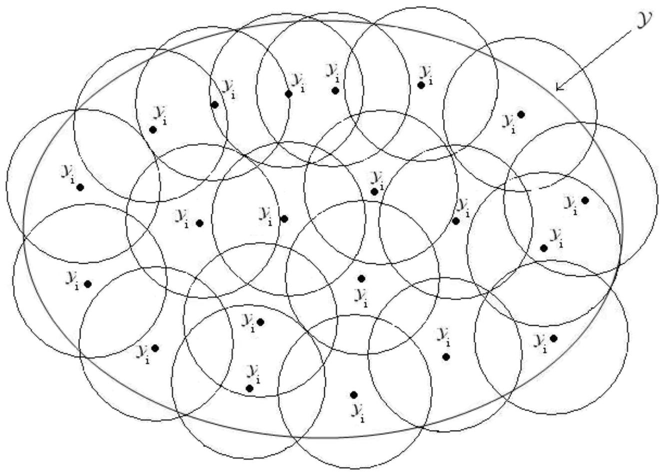

Figure 7: The N_1 balls of radius 1 which cover $\mathcal{Y}$, and their centers y_i ($i \leq N_1$).

$|x - y| < \delta$, x and y must be in the same subinterval of T_∞. So $f(x) = f(y)$, or $d(f(x), f(y)) < \epsilon$. Thus f_1 is continuous. Now we need to construct f_2. Consider y_a, one of the centers of the balls in our finite cover. Since $B_1(y_a)$ is an open set, for every $b \in B_1(y_a)$ there exists an $r_b > 0$ (which we can assume without loss of generality to be less than 1/2) such that $B_{r_b}(b) \subseteq B_1(y_a)$. Choose such an r for each element of every ball in our finite cover. From now on, we will refer to these rs as r_2, with the understood distinction that r_2 may be different for each individual element of $\mathcal{Y}$, but that r_2 is less than 1/2 for all y. Then $\{B_{r_2}(b) : \ b \in y_a\}_{a \leq 2^{m_1}}$ is an open cover for $\mathcal{Y}$[4]. Note: We will also write $r_1 = 1$, and in general, let r_n denote the radius of the nth-stage balls[5], which we will choose to be less than $(1/2)^{n-1}$.

[4]The reader should understand that the relevant question of notation is something of a double-edged sword. The duller edge, so to speak, is the problem of excessive notation, an issue we shall skirt shortly. The more subtle and more difficult question is how to refer to these collections of balls of different radii. We could keep track of every radius for every element, but that strategy would make the excessive notation problem unbearable. I tried $B_{r<0.5^{n-1}}$, but wording was too tricky without a clearly named radius. Professor Holdener suggested defining a family of balls, each matching the criteria we require; I prefer a more active approach, selecting each radius individually, as it were. It is my opinion that the notation I have chosen allows us to choose specific radii for each element of $\mathcal{Y}$, use the corresponding balls as open covers, and ignore the unnecessary details.

[5]We will say "balls of radius r_n" for ease of notation; as noted above we understand

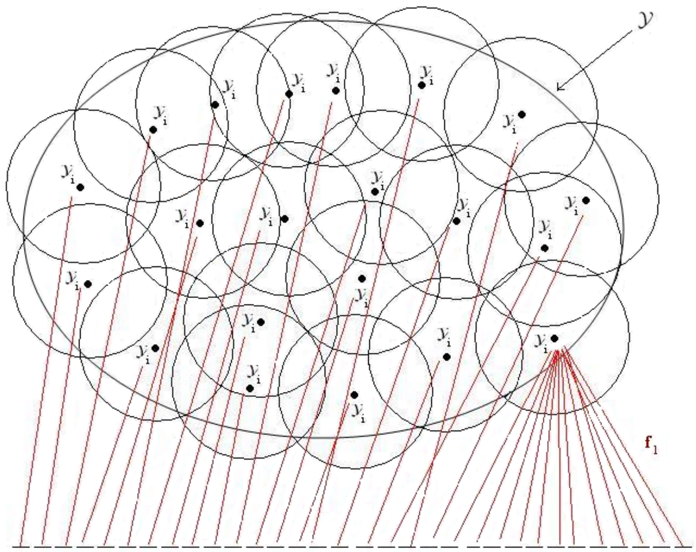

Figure 8: f_1 maps 2^{m_1} subintervals of the Cantor set to the N_1 centers of balls of radius 1 which cover $\mathcal{Y}$. Notice how f_1 illustrates our assumption that $N_1 = 2^{m_1}$.

Our next step is to take a finite subcover of the above collection, and to realize that every ball of radius r_1 will be covered by finitely many balls B_{r_2} (Figure 9). Of course, each ball of radius r_1 will likely require a different number of smaller covering balls, a fact we can express as follows: the second-stage finite subcover has N_2 elements, all balls of radius less than $1/2$ (radius r_2); each ball $B_1(y_i)$ requires $N_{2,i}$ smaller balls to cover (and the sum over i of $N_{2,i}$ equals N_2). We would then index the centers of these second-stage balls as $y_{j,i}$.

It becomes clear at this point that our notation is about to get very, very complicated, so we are not out of order in taking a few measures to simplify. Again by repeating centers as necessary, we can assume that $N_{2,i} = N_{2,j}$ for all i and $j \leq 2^{m_1}$. That is, we assume that we can cover each ball of radius r_1 with exactly the same number of balls of radius r_2. Using the same argument as above, we can assume without loss of generality that this number is a power of two, 2^{n_1}. If the reader wonders why we use n_1 and not m_2, the

that r_n is not necessarily constant over $\mathcal{Y}$.

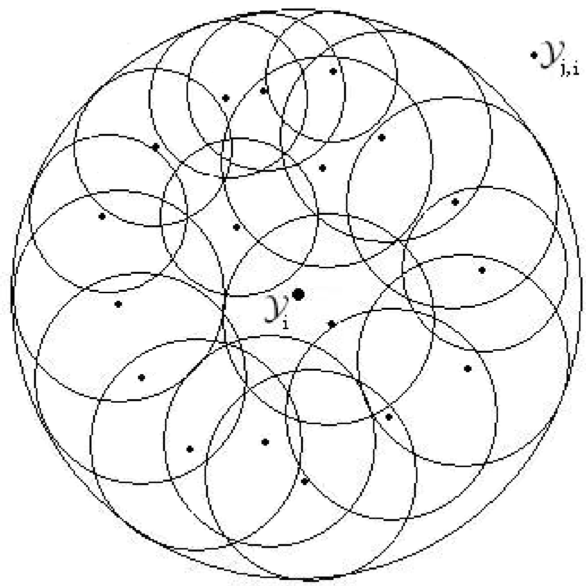

Figure 9: The $N_{2,i}$ balls of radius less than 0.5 which cover the ball of radius 1 around $\mathcal{Y}_i$, and their centers $\mathcal{Y}_{j,i}$ $(j \leq N_{2,i},\ i \leq N_1)$.

reason is this: we would like (m_k) to indicate the stages of the Cantor Set we will use to define the functions (f_k), but n_1 is how far down we jump into the Cantor Set from T_{m_1}. We'll need an entire sequence of jumps, so we'll let that be (n_k).

By now the reader may have anticipated our strategy. We took a finite cover and assumed it to have 2^{m_1} elements. We then used the subintervals of T_{m_1} to define a continuous function $f_1 : T_\infty \to \mathcal{Y}$. Now, for each ball in that first finite cover, we have a finite cover with 2^{n_1} elements. Notice that for $m_2 = m_1 + n_1$, T_{m_2} divides every subinterval of T_{m_1} into 2^{n_1} subintervals. We will use these subintervals to define $f_2 : T_\infty \to \mathcal{Y}$.

Let's consider an example to clarify before we define f_2. Fix $i \leq 2^{m_1}$. Then f_1 maps s_i to y_i. Now we have divided $B_1(y_i)$ into 2^{n_1} smaller balls with centers $y_{j,i}$, where $j \leq 2^{n_1}$. In T_{m_2} we have divided s_i into 2^{n_1} subintervals $s_{j,i}$. Then f_2 should map $s_{j,i}$ to $y_{j,i}$. Again, given any $x \in T_\infty$, there exist $j^\star$ and $i^\star$ such that $x \in s_{j^\star,i^\star}$. So define $f_2 : T_\infty \to \mathcal{Y}$ by:

$$f_2(x) = y_{j^\star,i^\star}.$$

Since f_2 is constant over each $s_{j,i}$, and these subintervals are disjoint, we can conclude again that f_2 is continuous.

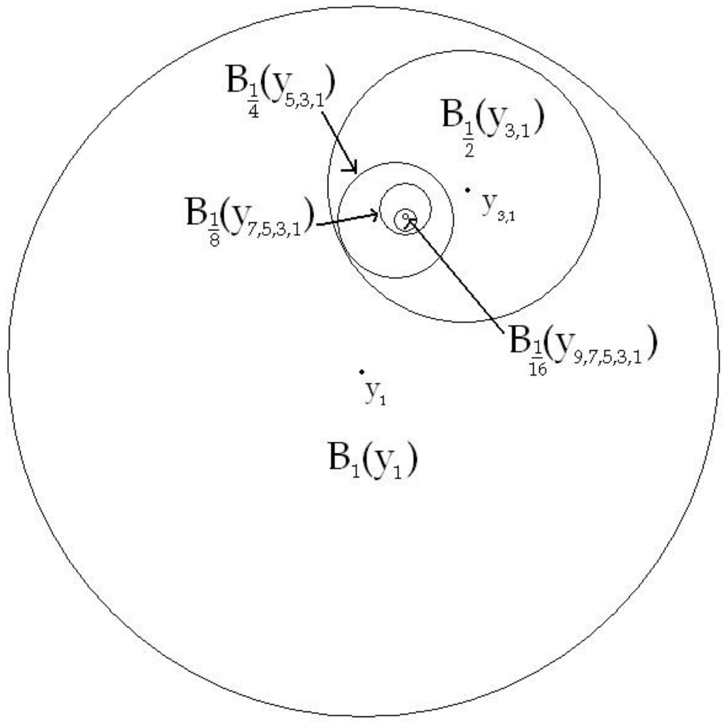

Figure 10: A sample sequence of stacked balls of decreasing radius.

Continuing this construction inductively, let us now assume that we have constructed the complete sequence of functions (f_n) using finite covers of balls of radius r_n, defining these functions from smaller and smaller subintervals of the Cantor Set to the centers of these balls. Now we should try for a little intuition. We hope to show that this sequence of functions converges to a continuous function which is onto $\mathcal{Y}$. But does this sequence of functions even converge? Well, the ranges of our functions are strongly linked to a sequence of balls whose radii decrease faster than 0.5^{n-1} (See Figure 10). Notice that the centers of a stacked sequence of balls (meaning smaller balls contained in larger balls) would have to be a Cauchy sequence because of this decreasing radius criterion. Fix $x \in T_\infty$. Since (f_n) maps x to the centers of a stacked sequence of balls, $(f_n(x))$ would have to be a Cauchy sequence. So given any ϵ, we should be able to find a point in our sequence after which any $f_n(x)$ and $f_m(x)$ would both be contained in the same tiny ball. That is, our sequence seems to be uniformly Cauchy. Let's prove this claim.

Lemma 2: This sequence (f_n) is a uniformly Cauchy sequence of functions.

Proof:
Fix $\epsilon > 0$ and choose any $x \in T_\infty$. Choose $N \in \mathbb{N}$ such that

$$\left(\frac{1}{2}\right)^{N-1} < \epsilon.$$

Now fix $p > q > N$. Notice that $f_q(x) = y_{i,j,k,\ldots,z,\alpha}$ for some $i \leq 2^{n_{q-1}},\ j \leq 2^{n_{q-2}}, \ldots, z \leq 2^{n_1}, \alpha \leq 2^{m_1}$. (The reader should be glad here that we skipped all that notation up to this point). Now we should also recognize that for all $m > q,\ f_m(x) \in B_{r_q}(f_q(x))$. That is, successive function values of x will be contained in the ball centered on $f_q(x)$. But

$$\left(\frac{1}{2}\right)^{q-1} < \left(\frac{1}{2}\right)^{N-1} < \epsilon.$$

Then

$$\begin{aligned} f_p(x) &\in B_{r_q}(f_q(x)) \\ &\subseteq B_\epsilon(f_q(x)). \end{aligned}$$

So $d(f_q(x), f_p(x)) < \epsilon$. Then (f_n) is a uniformly Cauchy sequence of continuous functions. ∎

Lemma 3: (f_n) converges to some continuous function $f : T_\infty \to \mathcal{Y}$.

Proof:
First, we notice that $\mathcal{Y}$ is complete because it is compact. Then by Theorem 11, we can conclude that $(f_n) \to f$, where $f : T_\infty \to \mathcal{Y}$ is a continuous function. ∎

It is this function f which we will show to be onto $\mathcal{Y}$.
Lemma 4: $f : T_\infty \to \mathcal{Y}$ is onto.

Proof:
Fix any $y \in \mathcal{Y}$. We must find the $\chi \in T_\infty$ such that $y = f(\chi)$. We know that there exists a y_i, where $i \leq 2^{m_1}$, such that $y \in B_1(y_i)$. Now recall that

$f[s_i] = y_i$. Because f maps elements of s_i to $B_1(y_i)$, we know our target χ is in s_i. So, in order to locate χ, we need a set of directions to get to s_i. Fortunately, we are well acquainted with elements of the Cantor Set and how to find them. Suppose that (a_n) is the ternary expansion sequence of some $a \in s_i$. Since a_1 tells us where to go in T_2, i.e. which subinterval to go to, a_{m_1-1} will tell us which subinterval of T_{m_1} to go to. That is, $(a_1, a_2, \ldots, a_{m_1-1})$ is the "address" of s_i. More importantly, these are the first $m_1 - 1$ terms of the expansion sequence for χ! We only need infinitely many more before we have the exact location of χ.

Now there must a $y_{j,i} \in B_1(y_i)$ such that $d(y, y_{j,i}) < 0.5$. Notice that $j \leq 2^{n_1}$, and that f maps the subinterval $s_{j,i}$ of T_{m_2} to $B_{r_2}(y_{j,i})$. Then $\chi \in s_{j,i}$. Again, we choose $b \in s_{j,i}$, and find its ternary expansion sequence (b_n). Of course, the terms up to b_{m_1-1} will just be the address of s_j; the terms through (b_{m_2-1}) give the address of $s_{j,i}$, and the next $m_2 - m_1$ terms in the expansion sequence for χ.

So this becomes our strategy: Since our open cover balls decrease in radius faster than $2^{-(n-1)}$, we find centers of balls that get closer and closer to y. Then we find the addresses (as a sequence of zeros and twos) of the subintervals which f maps to these centers. These addresses will give us a unique sequence of zeros and twos which define χ. We needn't repeat this set of instructions because they are the same at every step. Having "completed" this well-defined process, we may assume the result. Formally,

$$\chi = \sum_{n=1}^{\infty} \frac{\chi_n}{3^n},$$

where

$$\begin{aligned}(\chi_n) &= (\chi_1, \chi_2, \ldots, \chi_{m_1-1}, \chi_{m_1}, \chi_{m_1+1}, \ldots) \\ &= (a_1, a_2, \ldots, a_{m_1-1}, b_{m_1}, b_{m_1+1}, \ldots).\end{aligned}$$

In order to show that a function f is onto, given any point b in the codomain, one must produce a candidate a for which $f(a) = b$. This is the mathematician's chance to be as obscure as possible, for we are not required to explain the origin of a, and black magic is as good as trial and error, for in the end, all we need is $f(a) = b$. I hope, however, that we have been more clear concerning the origin of χ, its location in subintervals of T_∞, and its relationship to center points in our finite covers of $\mathcal{Y}$.

But as we have said, it is most important that $f(\chi) = y$. If there are any remaining questions about how we found χ, or what it is, the closing of this proof may clear them up. Recall that for any $x \in \mathcal{X}$,

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Then

$$f(\chi) = \lim_{n \to \infty} f_n(\chi).$$

We have said this before, but again we shall point out that $d(f_1(\chi), y) < 1$, $d(f_2(\chi), y) < 1/2$, $d(f_n(\chi), y) < (1/2)^{n-1}$, and so on. Then $(d(f_n(\chi)), y) \to 0$. Since the distance from $f_n(\chi)$ to y converges to zero, we can conclude that $y = f(\chi)$. Thus f is onto $\mathcal{Y}$. ∎

Finally, we can conclude that f is a continuous function from the Cantor Set onto $\mathcal{Y}$. ∎

So every compact metric space $\mathcal{Y}$ is the continuous image of the Cantor Set under some function! This is a very interesting result, and has a corollary that is far from obvious (before now).

11 Corollary and Conclusion

Suppose that $\mathcal{Z}$ is any compact metric space. Let f be the continuous function such that $f[T_\infty] = \mathcal{Z}$. Since f is onto $\mathcal{Z}$, we know that $card(\mathcal{Z}) \leq card(T_\infty)$. So $card(\mathcal{Z}) \leq card(\mathbb{R})$! There are no compact sets larger than the real numbers (or more accurately, with cardinalities greater than the cardinalities of the reals). At first, this may have seemed an obstacle to the proof of the theorem in §10.2, but in no part of that proof did we use any assumptions about the cardinality of $\mathcal{Y}$.

Finally, we have achieved our goals for this study of the Cantor Set. While, with the exception of the proof in §10.2, none of the individual theorems were particularly difficult, it was the continual juxtaposition of unusual and tricky concepts that provided the most obstacles to progress. While I have no perfect remedy for any of the reader's lingering bewilderment, I can say for sure that I always found a few good hours of staring at the ceiling and letting those concepts sink in to be rather helpful.

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