# The Jordan Curve Theorem:
# A Combinatorial Approach

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# 1 Introduction

The Jordan Curve Theorem is a fascinating piece of mathematics, which provides a rigorous proof for what appears to be a basic fact about topology. As with many things in mathematics, the Jordan Curve Theorem seems like a trivial statement, but turns out to be decidedly non-trivial to prove. For this reason, the Jordan Curve Theorem provides an excellent guide to an introduction to combinatorial topology. Several completely different methods to approach the Theorem exist, the most powerful of which allow one to extend the proof beyond the 2-dimensional case that we will consider. These methods are far beyond the scope of this paper, we will concentrate instead on developing the fundemenal tools of combinatorial topology, and then applying these ideas to the specific case of the Jordan Curve Theorem.

However, before even thinking about proving the Jordan Curve Theorem, one should know what it says. The statement of the theorem follows:

**Theorem 1. (Jordan Curve Theorem)** *Let $\mathfrak{J}$ be a Jordan curve. Then the compliment of $\mathfrak{J}$ in the plane, $\mathfrak{J}'$, is not connected but consists of two disjoint connected pieces, one of which is bounded and one of which is not bounded. The curve $\mathfrak{J}$ forms the boundary for both pieces.*[1]

First of all, there is the question of what is a Jordan curve. A more sophisticated definition will follow, but for right now, a Jordan curve is simply a closed curve, a loop in the plane. All that Theorem 1 says is that any closed loop in the plane will split the plane into two parts: an inside and an outside. In addition, these two parts are different, in that one is bounded by the curve (the inside), and one is unbounded (the outside). This seems straightforward enough, after all, draw any curve that connects back up to itself, and there you have it: an inside and an outside. In fact, dividing the plane seems almost the definition of what a closed curve does. However, consider something a bit uglier than any curve you could draw on a piece of paper, a space filling curve. These are curves that fill up the entire region over which they are defined. Well known examples include Peano curves, which can be constructed from iterating on simple curves. Now, the interesting thing is that any finite stage of this iteration will yield a Jordan curve that divides the plane, however, the limit of this sequence, the Peano curve itself, does not. The Jordan Curve Theorem gives a way of telling when such curves do and do not divide the plane. We shall return to this example at the conclusion, indicating at what point a space filling curve fails to be a Jordan curve.

A successful proof of the Jordan Curve Theorem is critical for mathematics. First of all, there is the obviousness of its truth. Mathematics should be able to confirm our intuition in cases like this. If it turns out that it cannot, there is something worrisome about our construction of the math. Second, the Jordan Curve Theorem is useful. In many cases in mathematics, you do need to be able to show that the inside of any given closed curve is in fact distinct from the outside. For example, in complex analysis, the entire concept of contour integrals requires the ability to distinguish between the inside and the outside of the

[1] [2] p.81

curve in question. Without the Jordan Curve Theorem, this idea would be impossible.[2]

## 2 Topology

In order to truly understand the Jordan Curve Theorem, we will need a more rigorous definition of a Jordan Curve. We can start by deciding what we would want a 'simple closed curve' to be. Clearly, a closed curve should be just that: closed. There should be no breaks in the curve, and no 'ends.' In addition, we do not want the curve to cross itself. Now let us consider what common figure is the epitome of what we want in a Jordan curve. The simplest that comes to mind is the unit circle. If we could stretch and bend the boundary of the unit circle without breaks or crossings, it seems reasonable to say that we could come up with any general simple closed curve we wish.

This section provides a mathematical basis for the argument in the previous paragraph. The basic problem that needs to be addressed is the idea of "bending and stretching" the unit circle without disturbing any of the properties that make the unit circle our archetypical Jordan curve to begin with. Basically, if two points are 'near' each other on the circle, then we want them to be near each other in the final curve. The area of mathematics that formalizes these ideas is topology, so we begin with some basic definitions.

The first thing we want to define is what it means for two regions of the plane to be 'near' each other. We start by defining a region around a point as a neighborhood.

**Definition 1. Neighborhood** *Let $p$ be a point in the plane. Then a neighborhood of $p$ is any open ball containing $p$.*

Now with this idea, we can move on to define what it is for two points to be near each other.

**Definition 2. Near** *Given a set $A$ and a point $p$ in the plane, $p$ is near $A$ (denoted $A \leftarrow p$) if every neighborhood of $p$ contains an element of $A$.*

Students of real analysis may recognize this definition as very similar to that of a limit point. For example, if we consider the sequence $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ which converges to zero, we see that 0 is near the set $\{a_n\}$. To prove this, we must show that any neighborhood of zero contains at least one point of the sequence. The usual notation for an open ball centered at a point $p$ with radius $r$ is $B_r(p)$. Let $\epsilon > 0$, then we wish to show that $B_\epsilon(0)$ contains a point of $\{a_n\}$. If the distance from an element of the sequence to zero is less then $\epsilon$, then that element is in the open ball. That is, since the $n$th element in the sequence is $\frac{1}{n}$, for $a_n$ to be in $B_\epsilon(0)$,

$$\begin{aligned} \left|\frac{1}{n} - 0\right| &< \epsilon \\ \frac{1}{n} &< \epsilon. \end{aligned}$$

[2] [5] p.110

Now it is a property of the real numbers that for all positive reals, there exists an integer $n$ such that $n^{-1}$ is less than that number. Therefore, there exists $n$ such that the previous inequality is true, and every neighborhood of zero contains at least one point in the sequence (actually it contains an infinite number, but that is unnecessary to prove in this case). As this example illustrates, we can consider nearness for a set, or as the limit point of a sequence.

Now, armed with these definitions, we next consider continuity of functions. From calculus or real analysis, one might recall the following definition:

**Definition 3. Continuity (I)** *Let $D$ and $R$ be subsets of the plane and $f : D \to R$ be a function. $f$ is continuous if, for every $a \in D$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $|b - a| < \delta$, $|f(b) - f(a)| < \epsilon$.*

Careful consideration of this definition in light of the previous definitions yields an interesting connection. All that Definition 3 is stating is that points and sets that are near in the domain of $f$ are near in the range. We can formalize this in another definition for continuity.

**Definition 4. Continuity (II)** *Let $f : D \to R$ be a function. $f$ is continuous if, for any $a \in D$ and $B \subseteq D$, $B \leftarrow a$ implies that $f(B) \leftarrow f(a)$.*

Of course, if we have two definitions for the same term, we better show they are the same thing. The following proves this equivalence.

**Theorem 2.** *Definitions 3 and 4 are equivalent.*

*Proof.*

($\Longrightarrow$) Let $f : D \to R$ be continuous(I). Let $a \in D$ and $B \subseteq D$ such that $B \leftarrow a$. Let $\epsilon > 0$. Since $f$ is continuous(I), there exists $\delta > 0$ such that if $|b - a| < \delta$ for $b \in D$, $|f(b) - f(a)| < \epsilon$. $B \leftarrow a$, so for all $\delta > 0$, there exists an open ball of radius $\delta$ centered at $a$ containing a point of $B$. Call this point $b_\delta$. Then $|b_\delta - a| < \delta$, so $|f(b_\delta) - f(a)| < \epsilon$. Since this is true for all $\epsilon > 0$, every neighborhood of $f(a)$ contains a point of $f(B)$, namely $f(b_\delta)$. Therefore $f(B) \leftarrow f(a)$. Thus, $f$ is continuous(II).

($\Longleftarrow$) Let $f : D \to R$ be continuous(II). Fix $\epsilon > 0$. Let $a \in D$ and $B \subseteq D$ such that $B \leftarrow a$. Since $f$ is continuous(II), $f(B) \leftarrow f(a)$. Therefore, for all $b \in B$ such that $|f(b) - f(a)| < \epsilon$. Since $B \leftarrow a$, every $b$ is in a neighborhood of $a$, so for each $b$ there exists $\delta_b > 0$ such that $|b - a| < \delta_b$. Hence, $f$ is continuous(I).

□

This theorem is very useful, though it may not seem so to begin with. Its use derives from our familiarity with the first definition of continuity: we know what a continuous function is in this sense. However, the second definition allows us to say whether a function transforming some set in the plane is preserving the nearness of points in that set, which we will soon see implies that the function preserves what are known as the topological properties of that set.

We can employ this definition to determine whether shapes in the plane possess similar qualities (such as a circle and a Jordan curve).

**Definition 5. Topological Transformation** *Let $D$ and $R$ be subsets of the plane. A function $f : D \to R$ that is continuous and invertible is a topological transformation. Two sets in the plane are topologically equivalent if there exists a topological transformation that maps one to the other and vica versa.*

Any quality of a set that is respected by topological transformations is called a topological property. That is, if some set $A$ has some property $x$, and all topological equivalents of $A$ have the property $x$ as well, then $x$ is a topological property. It should be clear that nearness is a topological property, by definition of topological transformation and continuous functions. In fact, this connection is so basic to the definition of the involved concepts that one can simply define a topological transformation solely in terms of nearness[3]

Let us now formally define a few basic shapes in the plane, including a Jordan Curve:

**Definition 6. Cell, Jordan Curve, and Simple Arc** *A set of points $\mathfrak{C}$ in the plane is a cell if $\mathfrak{C}$ is topologically equivalent to the closed unit circle $D = \{(x, y) | \sqrt{x^2 + y^2} \leq 1\}$. A set of points $\mathfrak{J}$ in the plane is a Jordan curve if $\mathfrak{J}$ is topologically equivalent to the unit circle $\{(x, y) | \sqrt{x^2 + y^2} = 1\}$. A set of points $\mathfrak{A}$ is a simple arc (also referred to as a path) if $\mathfrak{A}$ is topologically equivalent to the line segment* $[0, 1]$.

The question to ask at this point is whether this definition agrees with our intuition of what a Jordan curve should be. That is, given something that certainly should be a Jordan curve, can we find the topological transformation necessary to prove that it is? In most cases, for some arbitrary closed curve you draw, it would in fact be very difficult to determine the transformation, largely due to the problems of parameterizing the curve appropriately. However, for simple examples nicely described by a few equations, we can do so very easily.

Let us look at one of the simplest Jordan curves: the circle of radius two centered at the origin. We wish to determine a continuous, invertible function that maps the unit circle onto the larger circle. This is not very difficult, all we need to do is send every point on the circle to a point twice as far away from the origin in the same direction as the original point. Refer to Figure 1 for the picture of what the transformation is doing. The function can be expressed most conveniently in radial coordinates. In that system, $T[(r, \theta)] = (2r, \theta)$. Clearly this function is continuous, and invertible, $T^{-1}[(r, \theta)] = \left(\frac{1}{2}r, \theta\right)$. Therefore, the circle of radius two is a Jordan curve (as one would certainly hope).

Now let us consider something a bit more challenging: the semicircle in the last two quadrants (see Figure 2). Again, this is clearly a Jordan curve, but what function maps the unit circle to the semicircle? The unit circle can be described by the following equation:

$$1 = x^2 + y^2.$$

This can be solved for $y$ in terms of $x$ as

$$y = \pm\sqrt{1 - x^2},$$

[3] [4] p.46

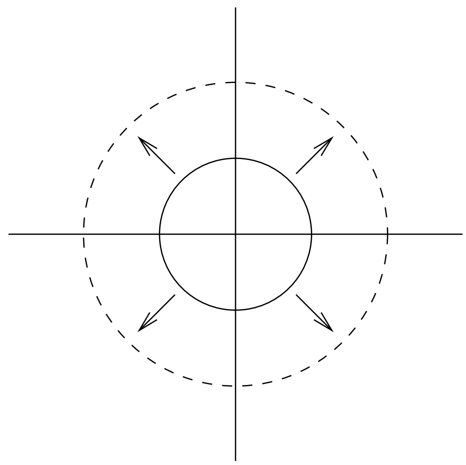

Figure 1: Unit Circle to Circle Radius 2

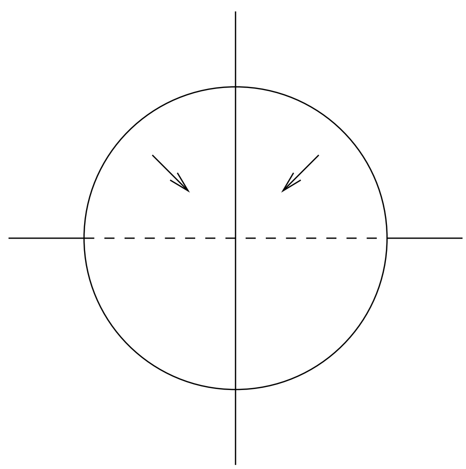

Figure 2: Circle to Semicircle

where the sign on the radical depends on whether you are in the upper or lower half of the plane. Now, we do not want to touch the lower half of the circle, and the upper half we simply want to map to the line segment connecting $(1,0)$ and $(-1,0)$, so our topological transformation is

$$T(x,y) = \begin{cases} (x,0) & \text{for} \quad y \geq 0 \\ (x,-\sqrt{1-x^2}) & \text{for} \quad y < 0 \end{cases}$$

where the domain of $T(x,y)$ is the unit circle. Now, this function is continuous, since the piece-wise definitions match up at the junctions $x = \pm 1$, but does it have an inverse? We see that the mapping we need to bring the hemisphere back to the initial unit circle is simple:

$$T^{-1}(x,y) = \begin{cases} (x,\sqrt{1-x^2}) & \text{for} \quad y = 0 \\ (x,-\sqrt{1-x^2}) & \text{for} \quad y < 0 \end{cases}$$

A little thought shows that $f^{-1} \circ f$ maps the unit circle back onto itself.

As one final example, let us show that an arc and a Jordan curve cannot be topologically equivalent. To do this, let us work with the archetypal examples of each: the unit circle and the unit line segment. If these two are topologically equivalent, then since all Jordan curves are topologically equivalent, as are all arcs, than all arcs and all Jordan curves would be equivalent to each other.

**Theorem 3.** *A simple arc is not topologically equivalent to a Jordan curve.*

*Proof.* Let us consider the unit circle $\mathfrak{C}$ and unit line segment $\mathfrak{L}$. Proceed by contradiction. Assume that there exists a continuous, invertible function $f : \mathfrak{L} \to \mathfrak{C}$. Consider the mapping of the endpoints of the line segment at 0 and 1. It is clear that, if we wish to preserve nearness of points inside the line as we transform it, then the only possible points that could connect to form a circle are the endpoints. That is, $f(0) = f(1)$ in order for $f(\mathfrak{L})$ to be $\mathfrak{C}$. However, since $f(0) = f(1)$, that means that $\{f(0)\} \leftarrow f(1)$, while it should be apparent that $\{0\} \not\leftarrow 1$. Thus, by the definition of continuity, $f$ cannot be continuous. So there does not exist a topological transformation from $\mathfrak{L}$ to $\mathfrak{C}$, since the unit circle and the unit line segment are not topological equivalent. See Figure 3 for an example. □

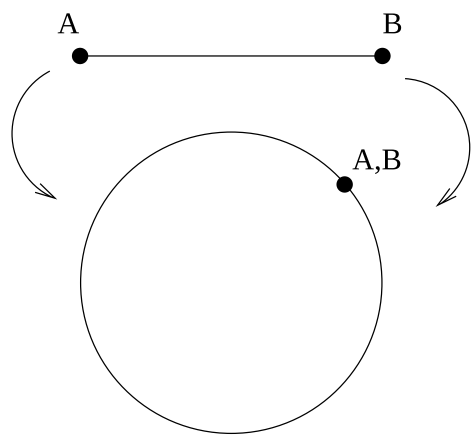


Figure 3: Arc Transformed to Circle

Once we have an idea of what Jordan curves are, we are ready to explore some of their topological properties. The end goal here is to come to a mathematical definition of what it means for two points in a set to be connected. As we shall see, this is a key idea in the proof of the Jordan Curve Theorem. To arrive at that definition will require a bit more background, much of which should be familiar from real analysis courses. First, let us remind ourselves of a few basic topological ideas about sets. The first concept, open sets, has been touched upon earlier in Definition 1, where we mentioned open balls. Formally,

**Definition 7. Open Set** *A set $A$ in the plane is open if, for all $a \in A$, there exists a neighborhood of $a$ entirely contained in $A$.*

Next, we have closure and closed sets,

**Definition 8. Closure, Closed Set, Boundary** *The closure of A, $\overline{A}$, is the set of all points p such that $A \setminus \{p\} \leftarrow p$. A set A is closed if $A = \overline{A}$. The boundary of A is the set of all points near both A and $A^c$.*

Another way of saying this is that if a set contains all of its near points (or limit points as they are also known), the set is closed. Students of real analysis may remember that if a set is open, the complement of the set is closed, and that the reverse is also true.

**Theorem 4.** *If A is open, then $A^c$ is closed. If A is closed, then $A^c$ is open.*

*Proof.* Let $A$ be an open set. Let $a \in \overline{A^c}$, the closure of the compliment. Since $A^c \setminus \{a\} \leftarrow a$, every neighborhood of $a$ contains a point of $A^c$ that is not $a$. Therefore, $a$ cannot be in $A$, since every element of $A$ has a neighborhood contained in $A$. Therefore, $a \in A^c$. Thus, $A^c = \overline{A^c}$, and is closed.

Now, let $B$ be a closed set. Let $b \in B^c$, therefore, $b \notin \overline{B}$, so not every neighborhood of $b$ contains a point of $B$. So there exists a neighborhood of $b$ entirely contained in $B^c$. This is true for all $b \in B^c$, so $B^c$ is closed.[4] □

One would like to be able to separate all sets into two disjoint categories: open or closed. It turns out you cannot do this. There are sets that are neither open nor closed, for example, an open ball of radius one and half of its boundary, the circle of radius one. The boundary in this case is part of the closure of the set. Since the set contains half of its boundary, it is not open, but since it doesn't contain the other half, it is not closed. There are also sets that are both open and closed, the plane and the empty set being the two most common examples. We see that, since every open ball about every point in the plane is in the plane, the plane is open. Additionally, every limit point of the plane must also be in the plane, so the plane is closed. For $\emptyset$, since it contains no elements, it is a vacuously true statement to say it contains all its limit points, or that every open ball about every element is contained in $\emptyset$, so it too is both open and closed.

Boundaries will be referred to later when proving the Jordan Curve Theorem. From our example with a half open ball, it seems logical that open sets should not contain their boundaries. We prove this here:

**Theorem 5.** *An open set does not contain its boundary.*

*Proof.* Let $A$ be an open set. Then the boundary of $A$, $b(A)$ consists of all the points near both $A$ and $A^c$. Let $b \in b(A)$. Then for all neighborhoods of $b$, there exists $x \in A^c$ such that $x$ is in the neighborhood of $b$. Therefore, there does not exist a neighborhood of $b$ entirely contained in $A$. Since $A$ is open, $b \notin A$. Therefore, $b(A)$ is not a subset of $A$. □

We now consider a subset of closed sets, compact sets. The definition we will use for compactness is technically known as sequentially compact. In general, compactness is a much more general concept

[4] [4] p.24

concerning open covers of a set[5], but we need not concern ourselves with that in this paper. Suffice to say, it can be shown that sequentially compact sets are compact, but not vica versa.[6] In this paper, we will use the terms sequentially compact and compact interchangeably, following the practice of Henle. So with that preface, we now define sequentially compact sets:

**Definition 9. Sequentially Compact** *A set $S$ is compact (sequentially compact) if every sequence $a_n \in S$ has a near point in $S$.*

Looking at Definitions 2 and 8, it should be clear that the requirements for compactness include the definition for closure. That is

**Theorem 6.** *If a set $S$ is compact, then it is closed.*

*Proof.* Let $S$ be a compact set $S$. Let $a \in \overline{S}$. Then there exists some set $A \in S$ such that $A \leftarrow a$. That is, for all $\epsilon > 0$, there exists a $b \in A$ such that $|b - a| < \epsilon$. We can therefore choose an $a_n \in A$ such that, for each $n \in \mathbb{N}$, $|a_n - a| < \frac{1}{n}$. The sequence of the $a_n$ converges to $a$, and therefore, $\{a_n\} \leftarrow a$. Since $S$ is compact, $a \in S$. Therefore, $S$ is closed. □

Now, if compact sets are closed, than are closed sets compact? It turns out that this is not true. For an example, consider a spiral out from the origin as shown in Figure 4

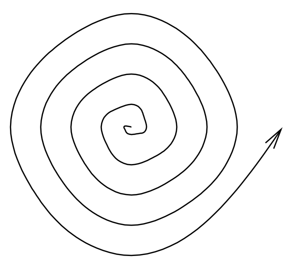

Figure 4: A Closed but not Compact Set

This spiral is closed, as can be seen clearly. Now, we can define a sequence along this spiral, the first point at the origin, the next point at some distance $\Delta$, and each following point twice the distance from the origin as the previous. This sequence is not near any point in the spiral, and therefore, the set cannot be compact.[7] From this, we can see that what we need to get compactness in addition to closure is some way of preventing the set from going off to infinity. That is, it must be bounded. There are several definitions of boundedness available in the literature, all of which are equivalent (as they must be). For our purposes, the following shall be used.

[5] [6] p.75
[6] [1] p.244
[7] [2] p.23

**Definition 10. Bounded** *A set $S$ is bounded if it can be contained in a rectangle.*[8] *Alternatively, a set is bounded if it can be contained in an open ball.*[9]

Now we can prove our previous declaration concerning the relation between closed, bounded sets and compactness. This theorem is known as the Bolzano-Weierstrass Theorem, and follows the structure given in Stillwell.[10]

**Theorem 7. (Bolzano-Weierstrass Theorem)** *A set is compact if and only if it is closed and bounded.*

*Proof.*

($\Longrightarrow$) Let $A$ be a compact set. By Theorem 6, since $A$ is compact, it is closed. We prove that it is bounded by contrapositive. Assume that $A$ is unbounded. Than for every open ball of radius $n$, there exists a point $a_r$ in $A$ that is outside the ball. Let $a_0 \in A$ be some point. Then we can construct a sequence $\{a_n\}$ such that $a_n \in A$ is outside of $B_n(a_0)$. This sequence does not converge in $A$, therefore, $A$ is not compact. Therefore, by contrapositive, if $A$ is compact, it is bounded.

($\Longleftarrow$) Let $A$ be a closed and bounded set. Let $\{a_n\}$ be a sequence in $A$. Now, since $A$ is bounded, we can enclose it in a rectangle. Bisect this rectangle. One half of this bisection must contain an infinite number of points of $\{a_n\}$. Bisect this half, and so on, letting the length of the sides of the bisected rectangles go to zero. The limit of these bisected rectangles is a point $p$. Every rectangle about $p$ contains an infinite number of elements of $\{a_n\}$. Therefore, for all $\epsilon > 0$, there exists $a_m \in \{a_n\}$ such that $|a_m - p| < \epsilon$. Hence $\{a_n\} \leftarrow p$. Since $A$ is closed and $\{a_n\} \in A$, it contains all its near points, so $p \in A$. Thus, $A$ is compact.

□

Now, we want to prove that all Jordan curves are compact. The best way to do so is to show that compactness is a topological property, as laid out in Definition 5.

**Theorem 8.** *All Jordan curves are compact.*

*Proof.* We begin by proving that compactness is a topological property. Let $D$ be a compact set, and $f : D \to R$ be a topological transformation. Let $\{b_n\} \in R$ be a sequence. Consider the sequence $\{f^{-1}(b_n)\} = \{a_n\} \in D$. By Definition 9, there exists $a \in D$ such that $\{a_n\} \leftarrow a$. Therefore, since $f$ is a topological transformation, it is continuous, so $f\{a_n\} \leftarrow f(a)$. But $f\{a_n\} = \{f \cdot f^{-1}(a_n)\} = \{b_n\}$, so $\{b_n\} \leftarrow f(a)$. Since $a \in D$, $f(a) \in R$, so $\{b_n\}$ has a near point in $R$. Hence, $R$ is compact.

Since compactness is a topological property, it suffices to show that a single Jordan curve is compact. Let us consider the archetypal Jordan curve, the unit circle. Clearly, the unit circle can be contained in a rectangle, so it is bounded. The unit circle is in the complement of the open ball of radius one $(B_1(0))^c$,

[8] [2] p.23
[9] [6] p.40
[10] [7] p. 6

which is closed. Furthermore, it is in the closed ball of radius one $\overline{B_1(0)}$, which is, as the name implies, closed. From real analysis, the intersection of closed sets is closed[11]. Clearly $(B_1(0))^c \cap \overline{B_1(0)}$ is the circle of radius one, so the closed ball of radius 1 is closed. Therefore, it is compact. □

Note in this theorem the power of topological transformations. Once we have shown that a given property is topological, the hard work has been done for us. We can then prove our theorem for a simple example and extrapolate to the family of topological equivalents. A similar theorem proves that paths are compact as well. We will not prove this here, however.

To prove the Jordan Curve Theorem we want to be able to say whether two points (one on the "inside" and one on the "outside") are not in the same set. The simplest way to show this is to draw a line between the two and show that it crosses the boundary set by the Jordan curve. The mathematical language we need here is that of connectedness. Of course as with much in mathematics, there are several definitions,

**Definition 11. Connected(I)** *Let $D$ be a set. $D$ is connected if it contains no proper nonempty subsets that are both open and closed.*[12]

Alternatively,

**Definition 12. Connected(II)** *Let $D$ be a set. $D$ is connected if whenever it is partitioned into two proper nonempty subsets, one of these contains a point near the other.*[13]

Definition 12 is perhaps easier to get one's mind around. It should make sense that if a set is connected, by dividing it into two pieces, the resulting sets should still be really close to each other. Of course, if we have two definitions for the same word, we had better show that they are equivalent.

**Theorem 9.** *Definitions 11 and 12 are equivalent.*

*Proof.*

($\Longrightarrow$) Let $D$ be a connected(I) set. Let $A$ and $B$ be a nonempty partition of $D$. Then, since neither $A$ nor $B$ are the empty set or equal to $D$, both are either open or closed. Consider $A$. $A$ is either open or closed. If $A$ is open, then $A \neq \overline{A}$. So there exists $x \in D$ such that $A \leftarrow x$, but $x \notin A$. So $x \in B$. Therefore $B$ contains a point near $A$. Alternatively, if $A$ is not open, it must be closed. Therefore, since $A^c = B$, $B$ is open. By the previous argument, $A$ contains a point near $B$. So $D$ is connected(II).

($\Longleftarrow$) Let $D$ be connected(II). Proceed by contradiction. Assume there exists a nonempty partition of $D$, $A$ and $B$, such that $A$ is open and closed. Now, since $B$ is the compliment of $A$ in $D$, $B$ must be both open and closed as well, by Theorem 4. Since $D$ is connected(II), we know that one of these subsets contains a point near the other. Assume, without loss of generality, that it is $A$. Then there exists

[11] [6] p.53
[12] [1] p. 211
[13] [2] p.25

$x \in A$ such that $B \leftarrow x$. Since $x$ is near $B$, but not in $B$, than $B$ cannot be closed. This contradicts our assumption, so $D$ is connected(I).

□

Now that we have a definition of connectedness, what good is it? First, we notice that connectedness is a topological property. This should be fairly clear, since Definition 12 is based solely on nearness. Here is the formal proof.

**Theorem 10.** *Connectedness is a topological property.*

*Proof.* Let $D$ be a connected set, and let $T : D \to R$ be a topological transformation. Let $R$ be partitioned into to nonempty subsets $A$ and $B$. Then there exists $E, F \subset D$ such that $E = \{a \in D | T(a) \in R\}$ and $F = \{b \in D | T(b) \in B\}$. One of these sets $E$ and $F$ contains a point near the other. Assume that it is $E$. Then there exists $x \in E$ such that $F \leftarrow x$. Since $T$ is a topological transformation, $T(F) = B \leftarrow T(x)$. But $T(x) \in A$, so one of the sets in $R$ contains a point near the other. □

Now it would be nice if connectedness fit the day-to-day usage of the word. When we think of two areas of the plane being "connected," the most natural definition is that you can draw a line between the two without crossing the boundaries of the areas. Fortunately for us, our definitions of connected allow us to make that same statement. Coached in more mathematical terminology, what we want is two sets to be connected if every pair of points in them could be linked by a path. This would obviously be a great help in proving the Jordan Curve Theorem, since it would allow us to show whether the inside and outside are connected simply by showing that paths between the two cross the curve. To be able to do this, we need to prove two things. The first is that paths are connected. The second is that two sets are connected if all their pairs have connected sets containing them.

**Theorem 11.** *Paths are connected.*

*Proof.* (This theorem follows closely the one given in Henle[14], it is given here for completeness and as an example in the method of bisection.)

Since we proved in Theorem 10 that connectedness was a topological property, we need to prove that some general path is connected, and the general case follows. Since we have our choice of any path, we choose the simplest, the unit line segment $[0, 1]$. This proof relies on induction of the method of bisection; basically slicing the segment into smaller and smaller pieces to arrive at the point we want.

Let $A$ and $B$ partition $[0, 1]$. We want to show that one of these contains a point near the other. Let us prove the base case of the method of bisection. Bisect the segment into two equal parts, $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Now, we want to select one of these bisections with endpoints in both $A$ and $B$. If one of the intervals meets that requirement, select it. Assume that neither do, for example, assume that $\{0, 1, \frac{1}{2}\} \in A$. Then since $B$ is non-empty, one of these intervals must contain a point $b$ of $B$. We can therefore consider the subinterval

[14] [2] p.26

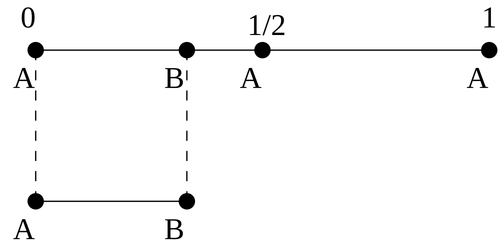


Figure 5: 1st Iteration of Method of Bisection

$[0, b]$. We can see an example of this "worst case scenario" in Figure 5. Since this argument is equally valid if $\{0, 1, \frac{1}{2}\} \in B$, we see that we can construct a new line segment $[a, b]$ that is a subset of the unit line segment with endpoints in both $A$ and $B$. Furthermore, since $[a, b]$ is a strict subset of $[0, 1]$, the length of the line segment is less than one. This ends the base case of the induction argument.

Let $[a_n, b_n]$ be a subset of the unit line segment with endpoints in both $A$ and $B$ such that $[a_n, b_n] \subset [a_{n-1}, b_{n-1}] \subseteq \cdots \subset [0, 1]$, all the intervals (with the possible exception of $[0, 1]$) having one endpoint in $A$ and the other in $B$, and each shorter than the previous. Bisect $[a_n, b_n]$. Then by an identical argument as in the previous paragraph, since $[a_n, b_n]$ contains elements of both $A$ and $B$ (at least the endpoints), we can select a subinterval $[a_{n+1}, b_{n+1}]$ that has endpoints in both $A$ and $B$. In addition, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n] \subset [a_{n-1}, b_{n-1}] \subseteq \cdots \subset [0, 1]$, so $|a_{n+1} - b_{n+1}| < |a_n - b_n|$. By the induction hypothesis, for all $n \in \mathbb{N}$ there exists $[a_n, b_n]$ such that

$$[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}] \subseteq \cdots \subset [0, 1]$$

with endpoints in both $A$ and $B$, and lengths tending toward zero.

It is pretty easy to get lost in the notation in that argument, so let us step back and consider what we have accomplished. We want to find a point in either $A$ or $B$ near the other set. We have now constructed a sequence in our path that contains an infinite sequence of elements of $A$ and $B$ (namely, the endpoints), each pair in sequence is getting closer together than the previous. Let $\mathcal{P}$ the the sequence of endpoints of our sequence of intervals that are in $A$. Since $[0, 1]$ is a path, it is compact. Therefore, by Definition 9 every sequence in $[0, 1]$ has a near point in $[0, 1]$. Therefore, there exists $P$ such that $\mathcal{P} \leftarrow P$. Similarly, we can define a sequence $\mathcal{Q}$ that consists of the endpoints of our sequence of intervals that lies in $B$. We claim that $\mathcal{Q} \leftarrow P$.

First, we know that $\mathcal{Q}$ must have a near point in $[0, 1]$ by compactness. Now, we want to prove that every open ball about $P$ contains a point in $\mathcal{Q}$. Fix $\epsilon > 0$. Now, since $\mathcal{P} \leftarrow P$, there exists an element $p_n \in \mathcal{P}$ such that $|p_n - P| < \frac{\epsilon}{2}$. Since the length of the line segments goes to zero, there exists $p_m \in \mathcal{P}$ such that $|p_m - q_m| < \frac{\epsilon}{2}$ (where $p_m$ are $q_m$ are the endpoints in $A$ and $B$ respectively). Let $s = \max\{m, n\}$. Then $|p_s - P| < \frac{\epsilon}{2}$ and $|p_s - q_s| < \frac{\epsilon}{2}$. Therefore, by the triangle inequality[15]

$$|q_s - P| \quad \leq \quad |q_s - p_s| + |p_s - P|$$

[15] [6] p.31

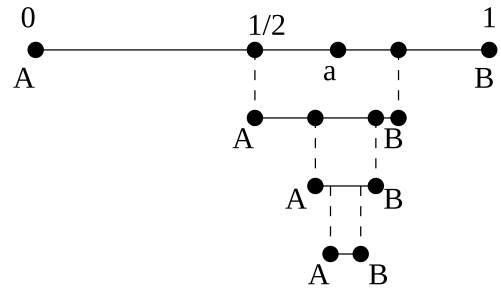


Figure 6: Three Iterations of Method of Bisection

$$\begin{aligned} &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ |q_s - P| &< \epsilon. \end{aligned}$$

Since this is true for all $\epsilon > 0$, every open ball about $P$ contains a point of $\mathcal{Q}$ Therefore, $\mathcal{Q} \leftarrow P$. Therefore, either $P \in A$ and is near $B$, or $P \in B$ and is near $A$. □

This proof may use concepts that are unfamiliar, so a moment should be taken to point out the important ideas. The essence of the method of bisection is to allow us to construct a sequence of intervals that get smaller and smaller while having "one end" in $A$ and the other in $B$. If $A = [0, a]$ and $B = [a, 1]$, then this sequence would converge down to the interface between them, namely $a$. The first few iterations of the bisection of this simpler example is shown in Figure 6. Once we have constructed the sequence of intervals, it is then straightforward to prove that there are points in one set that are near the other. This part of the theorem onward should be familiar to students of real analysis. In addition, remember that since connectedness is a topological property, we have shown all paths to be connected.

Now, we have one last thing to prove about connectedness. We wish to show that two sets are connected if every pair of points can be connected by a path. First, we show that a set is connected if every two points are contained in a connected subset.

**Theorem 12.** *If $S$ is a set such that, for any two elements of $S$, there exists a connected subset in $S$ containing those points, then $S$ is connected.*

*Proof.* Let $A$ and $B$ be a nonempty partition of $S$. We wish to show that one of these sets contains a point near the other set. Let $a \in A$ and $b \in B$. Then, by assumption, there exists a connected subset $C$ of $S$ that contains both $a$ and $b$. $C$ contains at least one point of both $A$ and $B$, so $A \cap C$ and $B \cap C$ are nonempty. Since $A$ and $B$ are disjoint and exhaust $S$, and $C \subseteq S$, $A \cap C$ and $B \cap C$ partition $C$. $C$ is connected, so one of these sets in $C$ contains a point near the other. Assume, without loss of generality, that $A \cap C$ contains a point near $B \cap C$. Then $A$ must contain a point near $B$. So $S$ is connected. □

As we will see in the next section, we will, in general, be dealing with collections of paths that can be divided up into straight lines that are perpendicular to a particular choice of Cartesian coordinates. These are not quite paths, since they allow for the constituent line segments to cross over each other. They are easier to work with, since we do not have to constrain them in that fashion. These constructs are called polygonal chains (polygonal because they consist only of straight lines, chains because they are set of linked straight lines)

**Definition 13. Polygonal Chain** *A polygonal chain is a sequence of a finite number of straight line segments parallel to the coordinate axes such that each line segment shares endpoints with adjacent line segments in the sequence.*[16]

Looking at a few examples in Figure 7, (a) is a polygonal chain, while the rest are not. The entire figure in (b) cannot be a polygonal chain, since it has a break in it. However, to two separate pieces both are polygonal chains. (c) is not a polygonal chain because, even though it is made up entirely of straight lines, those lines cannot all be parallel to the same set of coordinate axes. (d) cannot be a chain, since it has a curved section. One might wonder why we are interested in defining collections of line segments that are straight and mutually perpendicular. As seen in the next section, we will eventually overlay the plane with a grid, and at that point it becomes necessary to confine our lines to the edges inside this grating. Therefore, to avoid problems later on, we constrain the types of figures that we are interested in at this stage.

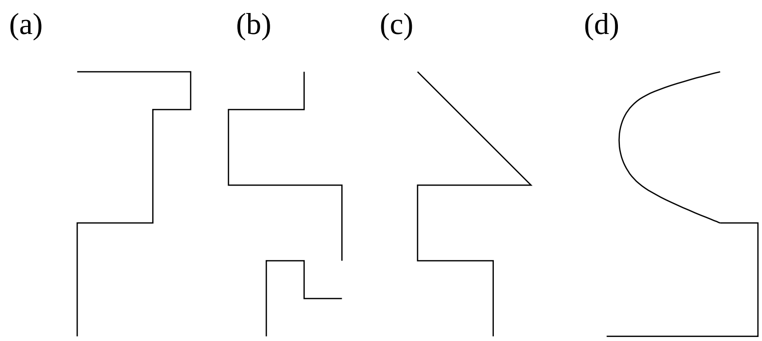


Figure 7: Examples of Chains and Things that are not Chains

We have proven that a set is connected if any two points in the set are in a connected subset. We have also proven that any path is connected. We would like to be able to prove that a set is connected iff any pair of points in the set can be connected by a path. Since we will be dealing exclusively with polygonal chains later on, we shall prove this theorem in terms of that subset of paths.

**Theorem 13. (Polygonal Chain Theorem)** *Let $G$ be an open set. Then $G$ is connected iff every pair of points in $G$ can be connected by a polygonal chain in $G$.*

*Proof.*

[16][2] p.81

($\Longrightarrow$) Suppose $G$ is connected. Let $a, b \in G$. We wish to show that there exists a polygonal chain in $G$ that connects $a$ and $b$ (that is, one endpoint of the chain is $a$, the other is $b$). Proceed by contradiction. Assume that there exists $a, b \in G$ such that $a$ and $b$ cannot be connected by a polygonal chain. We can therefore divide $G$ into two subsets. $A \subset G$ is the set of all points that can be connected by a polygonal chain to $a$, and $B \subset G$ is the set of all points in $G$ that cannot be connected by a polygonal chain to $b$. Clearly, $a \in A$ and, by assumption, $b \in B$, so these subsets are nonempty. Now, since $G$ is connected, by Definition 12, either $A$ or $B$ contains a point near the other.

Let us assume that $A$ contains a point $p$ such that $B \leftarrow p$. Then, since $G$ is open, there exists an open ball entirely contained in $G$ centered on $p$. By Definition 2, this open ball contains a point $r$ of $B$. Now, it is clear that if two points are inside an open ball, than there exists a polygonal chain between the two also inside the ball. Formally, we see that, if $p$ is placed at the origin (for convenience), the ball has radius $\epsilon$, and $r$ is at coordinates $(x, y)$ (where $\sqrt{x^2 + y^2} < \epsilon$), then there is a straight path from $(0, 0)$ to $(x, 0)$ which is entirely inside the ball, and another straight path from $(x, 0)$ to $(x, y)$ inside the ball. Therefore, these two paths, as they share endpoints and are parallel to the coordinate axes, form a polygonal chain connecting $p$ and $r$. Since the open ball is in $G$, the chain is in $G$.

Now $p \in A$, so there is a polygonal chain in $G$ connecting $a$ and $p$. We have shown that there is a polygonal chain in $G$ connecting $p$ and $r$, where $r \in B$. Therefore, combining these two chains, there is a new polygonal chain in $G$ connecting $a \in A$ and $r \in B$. A parallel argument can be made if $B$ contains a point near $A$. However, $B$ is the set of all points of $G$ that cannot be connected to $a$ via a polygonal chain. This is a contradiction, so every pair of points in $G$ can be connected by a polygonal chain in $G$.[17]

An example of this proof can be seen in Figure 8. Notice that the desired contradiction is not that we show that $a$ and $b$ can be connected, but that the set cannot be divided into two sections that cannot be connected. Clearly, if $B$ contains only $b$, then our proof shows that $a$ and $b$ are connected, but we need not assume that.

($\Longleftarrow$) If there is a polygonal chain connecting every pair of elements in $G$, then by Theorem 12, $G$ is connected, since a polygonal chain consists of a finite number connected sets, all of which are connected to each other.

$\square$

It turns out with the information we now have, we can prove the Jordan Curve Theorem for some simple curves. Take for example, the rectangle, with one corner at $(0, 0)$, and the far corner at $(a, b)$. To show the first part of the Jordan Curve Theorem: that the rectangle divides the plane into two unconnected sections, all we need to do (by Theorem 13) is show that there exists a point inside the rectangle that cannot be connected by a path to a point outside of the rectangle by a polygonal chain.

[17] [2] p.82

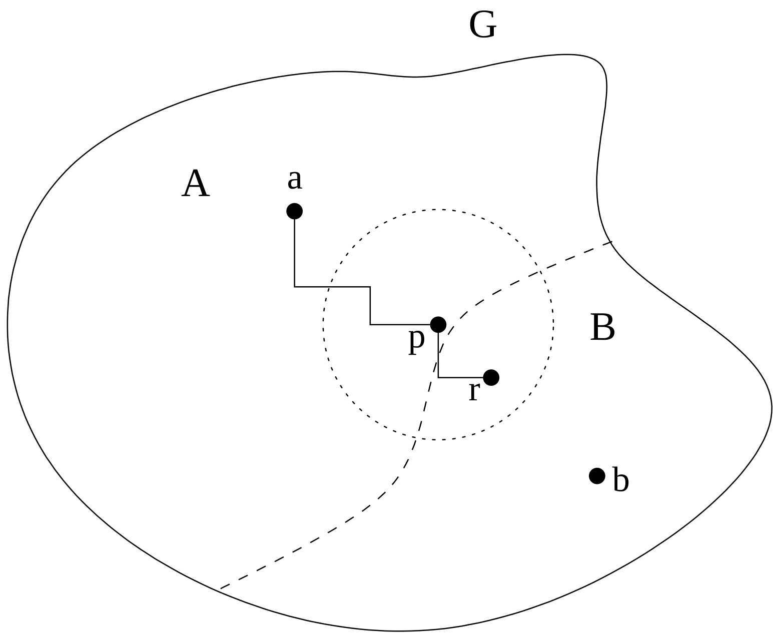


Figure 8: Polygonal Chain Theorem Exemplified

Let $(c, d)$ be a point inside the rectangle, and $(e, f)$ be a point outside. Let these two points be connected by a polygonal chain. There are four possibilities for the location of $(e, f)$. It is either above the rectangle $(f > b)$, below the rectangle $(f < 0)$, to the left of the rectangle $(e < 0)$, or to the right of the rectangle $(e > a)$. If it is to the right or above, any polygonal chain must pass through the right wall or the top of the rectangle, by the intermediate value theorem of calculus.[18] This is also true if the point lies to the left or below. Therefore, we see that the rectangle divides the plane into two non-connected sections. Clearly, the inside is bounded, since it is contained in a rectangle. The outside is not bounded, since any rectangle will fail to contain a least some points of the plane.

With one specific Jordan curve shown to satisfy the Jordan Curve Theorem, it is tempting to say that we are now done. After all, connectedness is a topological property, and all Jordan Curves are topologically equivalent, so if we prove it for one, we've proved it for all, right? Unfortunately, we only know that Jordan curves are topologically equivalent, not the spaces they divide the plane into. If you consider the Jordan Curve Theorem, you see that it is really a statement about the compliment of the curve, that is the plane that the curve divides. To prove the more general Jordan Curve Theorem, we need to do what we did with the rectangle: show that points "inside" the curve cannot be connected by a path to points "outside" the curve.

To do this, we need some more mathematical machinery. We need to be able to determine if a general path crosses another, and if it does, whether it is possible to "divert" it around the other curve. The tool that allow us to do this is algebra of chains on a grating, the subject of our next section. The drawback to

[18] [3] p.74

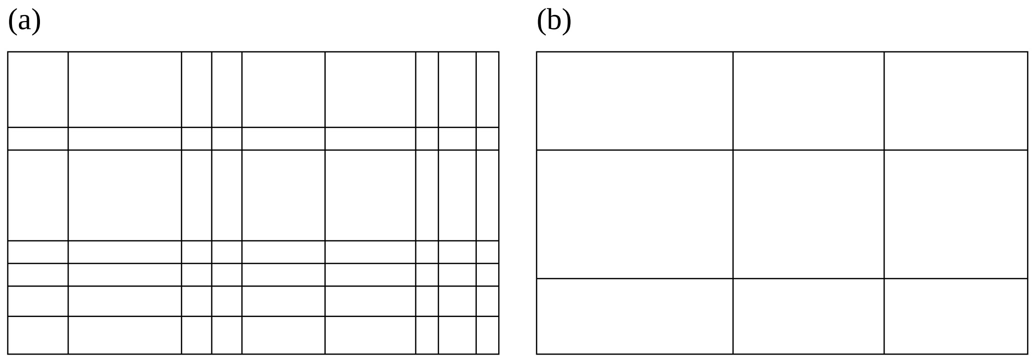


Figure 9: Sample Gratings

using this method in building a foundation for the Jordan Curve Theorem is that it allows us to work only with polygonized curves, that is, curves that can be drawn as a collection of lines at right angles. This works well enough for most normal uses, but excludes the really pathological examples of Jordan curves shown in the introduction.

## 3 Chains and Gratings

This section covers methods of dividing a closed rectangle in the plane into a grating of lines, and the useful theorems that arise. The basic concept is a grating, a closed rectangle, that for convenience we usually center on the origin with the borders aligned with the $x$ and $y$ axes. This is mostly so that we can speak colloquially about "above," "below," to the "right" of the grating and have it make sense.

**Definition 14. Grating, Simplex** *A grating* $\mathfrak{G}$ *is a rectangular set in the plane with a finite number of lines perpendicular to the borders. These lines define simplexes of* $\mathfrak{G}$*. Simplexes are sorted according to their dimension. A 0-simplex is a vertex, the intersection of two lines. A 1-simplex is a line segment of the grating between two 0-simplexes (that is, an edge of the grating). A 2-simplex is a face of the grating as defined by four 1-simplexes. The compliment of the entire grating in the plane counts as an additional 2-simplex.*

As an example, consider the gratings in Figure 9. In (a), we see that there are 80 0-simplexes, 142 1-simplexes, and 64 2-simplexes (counting the compliment of the outer rectangle). An easier example is (b), which has 16 0-simplexes, 24 1-simplexes, and 10 2-simplexes.

Why do we introduce this concept? Gratings can be placed over an area of the plane of interest, and they allow us to quantify sets and curves on the plane by counting their intersections with the $k$-simplexes of the grating ($k = 0, 1, 2$). Gratings therefore, allow us to do counting arguments with sets.

This idea of counting with simplexes is carried further with the introduction of sets of simplexes, called chains. We consider a chain to be a set of simplexes all of the same dimension. Therefore, a chain must be specified to be some $k$-chain, where $k$ is the dimension of the component simplexes. Some simplexes are

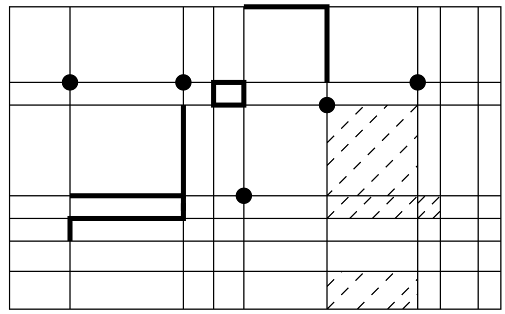

Figure 10: Sample $k$-chains

shown in Figure 10. Here, the indicated vertices constitute a 0-chain, the darkened edges are a 1-chain, and the lined faces are a 2-chain. Notice that being a chain does not require the elements to be "connected" in any sense. We see that there is a connection between the 1-chains that we have here and polygonal chains, they are both made up of a finite number of line segments parallel to the axes. The difference is that 1-chains do not have to consist of connected line segments.

We can now define the binary operation of "addition" on our $k$-chains.

**Definition 15. Chain Addition** *Let $\mathfrak{G}$ be a grating, and $C_1$ and $C_2$ be $k$-chains on $\mathfrak{G}$. Then the chain sum of $C_1$ and $C_2$, $C_1 + C_2$, is all $k$-simplexes in $(C_1 \cup C_2) \setminus (C_1 \cap C_2)$.*

That is, a $k$-simplex is in the sum if it is in $C_1$ or $C_2$, but not both. Chain addition defines three Abelian groups on $\mathfrak{G}$, one for each dimension of chain. The formal proof for this claim follows.

**Theorem 14.** *Let $\mathfrak{G}$ be a grating. The set of $k$-chains on $\mathfrak{G}$ (where $k = 0, 1, 2$) and the binary operation $+$ as defined in Definition 15 forms an Abelian group in $\mathfrak{G}$.*

*Proof.* To prove that a set with some binary operation is an Abelian group, we must prove closure, commutativity, associativity, the existence of an identity, and the existence of an inverse. Closure follows from the definition of chain addition, since the sum of two chains is again a chain.

To prove commutativity, we need to show that $C_1 + C_2 = C_2 + C_1$. Let $\mathfrak{G}$ be a grating, and $C_1$ and $C_2$ be $k$-chains in $\mathfrak{G}$. Chain addition deals with elements of sets, so to prove equality, an element proof can be used. Let $s$ be a simplex in $C_1 + C_2$. Then $s \in C_1 \cup C_2$, but $s \notin C_1 + C_2$. Therefore, $s$ is in exactly one of $C_1$ and $C_2$. Assume $s \in C_1$, then $s \notin C_2$. Therefore, $s \in C_2 \cup C_1$ but not $C_2 \cap C_1$, so $s \in C_2 + C_1$. If $s \in C_2$, a parallel argument follows. Therefore,

$$C_1 + C_2 \subseteq C_2 + C_1.$$

To prove inclusion in the other direction, an identical argument can be used. Therefore, chain addition is commutative.

To prove associativity, consider $k$-chains $C_1$, $C_2$, and $C_3$ in a grating $\mathfrak{G}$. We wish to prove that

$$C_1 + (C_2 + C_3) = (C_1 + C_2) + C_3.$$

Let $s$ be a $k$-simplex in $C_1 + (C_2 + C_3)$. Then $s \in C_1 \cup (C_2 + C_3)$ and $s \notin C_1 \cap (C_2 + C_3)$. Therefore, $s \in C_1$ or $s \in C_2 + C_3$.

- $(s \in C_1)$

  Assume $s \in C_1$, then $s \notin C_2 + C_3$, so either $s \notin C_2 \cup C_3$, or $s \in C_2 \cup C_3$ and $s \in C_2 \cap C_3$. Now, assume that $s \notin C_2 \cup C_3$, so $s \notin C_2 \cap C_3$. Therefore, $s \in C_1 \cup C_2$, but $s \notin C_1 \cap C_2$. Therefore, $s \in C_1 + C_2$. In addition, $s \notin C_3$, so $s \in (C_1 + C_2) \cup C_3$, but $s \notin (C_1 + C_2) \cap C_3$. Therefore $s \in (C_1 + C_2) + C_3$. Now let us go back and assume that $s \in C_1$, $s \in C_2 \cup C_3$, and $s \in C_2 \cap C_3$. Therefore, $s \in C_1, C_2, C_3$. Since $s \in C_1$ and $s \in C_2$, $s \notin C_1 + C_2$, since it is in the intersection of the two. However, since $s \in (C_1 + C_2) \cap C_3$, $s \in (C_1 + C_2) + C_3$.

- $(s \in C_2 + C_3)$

  Assume $s \in C_2 + C_3$. Then $s \notin C_1$, and $s \in C_2 \cup C_3$ but $s \notin C_2 \cap C_3$. Therefore, $s \in C_2$ or $s \in C_3$. Assume that $s \in C_2$. Then $s \in C_1 + C_2$, and therefore $s \in (C_1 + C_2) + C_3$. If $s \in C_3$, then $s \notin C_1 + C_2$, and therefore, $s \in (C_1 + C_2) + C_3$, since it is in the union, but not the intersection. This proves on direction of equality, namely

  $$C_1 + (C_2 + C_3) \subseteq (C_1 + C_2) + C_3$$

  We can then use the commutativity of chain addition to see that $(C_1 + C_2) + C_3 = C_3 + (C_1 + C_2)$. Then, going through our previous argument suffices to show that

  $$(C_1 + C_2) + C_3 \subseteq C_1 + (C_2 + C_3),$$

  thus proving equality, and therefore, the associative property.

We now want to prove the existence of an identity $k$-chain 0, such that, for any $k$-chain $C \in \mathfrak{G}$, $C + 0 = C$. Since chain addition removes any simplex that is in both chains being added, in order to remove no simplexes, we must add the empty set. Therefore, we wish to prove that, if $0 = \emptyset$, $C + 0 = C$. Let $s \in C + 0$. Then $s \in C \cup 0$ and $s \notin C \cap 0$. Since $0 = \emptyset$, if $s \in C \cup 0$, $s \in C$. Therefore, $C + 0 \subseteq C$. Now, let $s \in C$. Again, since $0 = \emptyset$, $s \notin 0$. Therefore, $s \in C \cup 0$ but not $C \cap 0$, so $s \in C + 0$. Therefore, $C \subseteq C + 0$, and we have proven equality, and thus the existence of an additive identity.

Finally, we wish to prove the existence of inverses. Let $C$ be a $k$-chain in $\mathfrak{G}$. We wish to find $-C$ such that $C + (-C) = 0$. That is, adding $-C$ must remove every simplex that was in $C$, without adding in any more. It should be clear that $-C = C$. Therefore we wish to show $C + C = 0$, and therefore $C$ is its own inverse. Let $s \in C$. Then $s \in C \cup C$, but $s$ is also in $C \cap C$. Hence, $s \notin C + C$. This is true for all $s \in C$. Now, let $s \notin C$. Then $s \notin C \cup C$, so $s \notin C + C$. It follows that, $C + C$ contains no simplexes of $\mathfrak{G}$, so $C + C = 0$. This concludes the proof that the set of $k$-simplexes on $\mathfrak{G}$ and chain addition forms an Abelian group. □

The previous proof was included to provide a feel for how chain addition works. Notice that chain addition basically counts whether a simplex is in an even or odd number of the chains being summed over. This is most noticeable in the portion of the proof dealing with associativity. If a simplex was in only one of $C_1, C_2, C_3$, than it was in the sum, likewise if a simplex was in all three chains. However, if the simplex was in exactly two of the chains, it is not in the sum. Therefore, it should be intuitively clear that if a simplex is in an odd number of chains to be summed over, it is in the sum, while if it is in an even number of the chains, it is not in the sum.

**Theorem 15.** *Let $\mathfrak{G}$ be a grating, and $\{C_i\}$ be k-chains in $\mathfrak{G}$. Then a simplex $s$ is in $\sum_i C_i$ if and only if $s$ is in an odd number of the $C_i$.*

*Proof.*

($\Longrightarrow$) We shall prove this direction by induction on $i$. We have seen from the associativity section of Theorem 14 that if $i = 2$, $s \in C_1 + C_2$ iff $s$ is in exactly one of $C_1$ and $C_2$. This proves the base case. Let $s \in \sum_{i=1}^{n} C_i$ and $s$ be in an odd number of the $C_i$. We wish to show that if $s \in \sum_{i=1}^{n+1} C_i$, $s$ must be in an odd number of the $C_i$.

Let $s$ be in an odd number of elements of the set $\{C_i | 1 \leq i \leq n+1\}$, that means that $s \notin C_{n+1}$. To show that this is true, let $s \in \sum_{i=1}^{n+1} C_i = \sum_{i=1}^{n} C_i + C_{n+1}$. Then $s$ is either in an odd number of $\{C_i | 1 \leq i \leq n\}$ and $s$ is not in $C_{n+1}$, or $s$ is in an even number of $\{C_i | 1 \leq i \leq n\}$ and $s$ is in $C_{n+1}$. By our assumption, $s$ is in an odd number of $\{C_i | 1 \leq i \leq n\}$, therefore, $s \notin C_{n+1}$. Hence, $s \in (\sum_{i=1}^{n} C_i) \cup C_{n+1}$ and $s \notin (\sum_{i=1}^{n} C_i) \cap C_{n+1}$, so if $s$ is in an odd number of $\{C_i | 1 \leq i \leq n+1\}$, $s \in \sum_{i=1}^{n+1} C_i$.

($\Longleftarrow$) We shall do this direction of the proof by contrapositive. That is, we shall prove that if $s$ is in an odd number of $C_i$, then $s \in \sum_i C_i$ by showing that if $s \notin \sum_i C_i$, $s$ is in an even number of the $C_i$. Again, we proceed by induction. The base case has again been proved in Theorem 14. Now, let $s \notin \sum_{i=1}^{n} C_i$ and $s$ in an even number of the $C_i$.

If $s$ is in an even number of elements of the set $\{C_i | 1 \leq i \leq n\}$, then for $s$ to be in an even number of elements of the set $\{C_i | 1 \leq i \leq n+1\}$, that means that $s \notin C_{n+1}$. Therefore, $s \notin \sum_{i=1}^{n+1} C_i = \sum_{i=1}^{n} C_i + C_{n+1}$ implies that either $s \notin (\sum_{i=1}^{n} C_i) \cup C_{n+1}$ or $s \in (\sum_{i=1}^{n} C_i) \cap C_{n+1}$. However, we know that $s \notin \sum_{i=1}^{n} C_i$, so $s$ cannot be in $(\sum_{i=1}^{n} C_i) \cap C_{n+1}$. Therefore, $s \notin (\sum_{i=1}^{n} C_i) \cup C_{n+1}$, implying that $s \notin C_{n+1}$. Therefore, $s$ is in an even number of the $C_i$. By induction, this is true for all $i$. By contrapositive, this proves that if $s$ is in an odd number of $C_i$, $s \in \sum_i C_i$.

□

Hopefully at this point, the idea of chain addition is understandable. Its usefulness to proving the Jordan Curve Theorem will appear later on, but as a quick preview, one use of these concepts is our ability to “cut out” overlapping regions by chain addition. Eventually, we will want to know whether a polygonal chain shares a boundary with a Jordan Curve, which we will approximate by a 1-chain. By adding the curve and the chain, we have a mathematical way of seeing whether they share any edges of an overlaid grating, which we can then work around.

It would be nice if we could find some way of distinguishing those 1-chains that are closed, that is, the ones in which the "first" segment shares an endpoint with the last. The way we do this for $k$-chains in general is to consider the boundary of the chain. We introduce the boundary operator $\partial$. The operator $\partial$ relates the addition we can do on $k$-chains with the algebra of $(k-1)$-chains. Clearly, this will break down in a sense for both $k = 0$ and $k = 2$ chains, since we don't have $(-1)$-chains or 3-chains, but we can fix those problems by some imaginative redefinitions.

Before we formally define the boundary operator of a chain, consider what we would want the boundary of a 2-chain to be. Looking at Figure 11 we see a 2-chain on the left, and what we would naturally assume to be the boundary of that chain on the right. As always, we want to have a mathematical definition that will preserve what our intuition tells us is true.

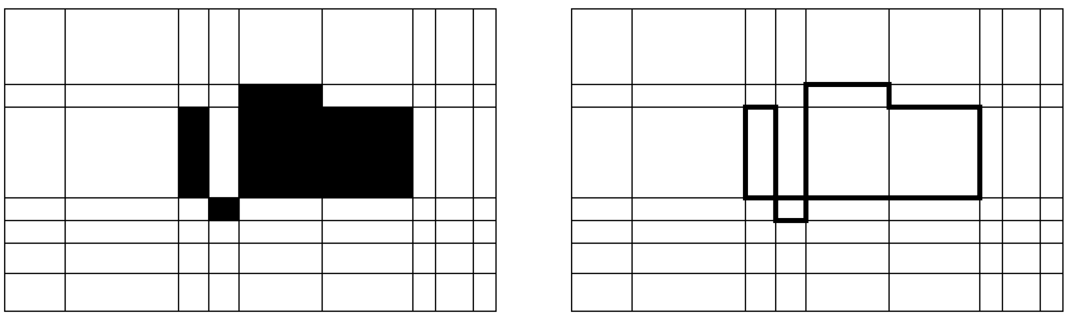

Figure 11: Boundary of a 2-chain on a Grating

Consider carefully why the lines inside the group of faces in Figure 11 are not in the boundary. Clearly, they should not be, since they are the edges of two faces. This gives us a handle on what it is to be a boundary: edges that are in two faces of the 2-chain should not be in the boundary. Expanding this idea to all $k = 1, 2, 3$,

**Definition 16. Boundary Operator** *Let $\mathfrak{G}$ be a grating, and $C$ be a $k$-chain in $\mathfrak{G}$. The boundary operator acting on $C$, denoted $\partial(C)$, consists of those $(k-1)$-simplexes that are in an odd number of the $k$-simplexes of $C$.*

A $k$-chain is a boundary if there exists some $(k+1)$-chain whose boundary is the $k$-chain. We get around the problem of what is the boundary of a 0-chain by defining it to be the empty set. Additionally, we say that no 2-chain other than the empty set is a boundary, since we will not deal with 3-chains in this paper. Notice also that our definition relies on the evenness or oddness of the number of simplexes. This should immediately suggest a connection to our chain addition. In fact, our whole concept of chain addition is pretty much defined to give us the following handy property of the boundary operator.

**Theorem 16.** *Let $\mathfrak{G}$ be a grating, and let $C_1$ and $C_2$ be k-chains in $\mathfrak{G}$. Then $\partial(C_1 + C_2) = \partial(C_1) + \partial(C_2)$.*

*Proof.* Again, we are dealing with chain addition, so we proceed by an element argument, where $(k-1)$-simplexes are our elements.

($\subseteq$) Let $d$ be a $(k-1)$-simplex in $\partial(C_1 + C_2)$. Then $d$ is in an odd number of $k$-simplexes of $C_1 + C_2$. Let the $k$-simplexes in $C_1 + C_2$ containing $d$ be $\{D_i | 1 \leq i \leq j\}$. It follows that $j$ is odd. For each $D_i$, $D_i$ can be in either $C_1$ or $C_2$, by Theorem 15. Therefore, $C_1$ contains a subset of $\{D_i | 1 \leq i \leq j\}$ called $E$ and $C_2$ contains a subset of $\{D_i | 1 \leq i \leq j\}$ called $F$. Since $E$ and $F$ partition $\{D_i | 1 \leq i \leq j\}$, $|E| + |F| = j$. Since $j$ is odd, either $|E|$ is odd and $|F|$ is even, or vica verse. Assume, without loss of generality, that $|E|$ is odd. This means that $d$ is contained in an odd number of $k$-simplexes in $C_1$, so $d \in \partial(C_1)$. Similarly, $|F|$ being even implies that $d$ is in an even number of $k$-simplexes in $C_2$, so $d \notin \partial(C_2)$. Since $d \in \partial(C_1) \cup \partial(C_2)$, but $d \notin \partial(C_1) \cap \partial(C_2)$, $d \in \partial(C_1) + \partial(C_2)$.

($\supseteq$) Let $d \in \partial(C_1) + \partial(C_2)$. Then $d$ is in either $\partial(C_1)$ or $\partial(C_2)$, but not both. Assume, without loss of generality, that $d \in \partial(C_1)$. Then $d$ is in an odd number $i$ of $k$-simplexes of $C_1$, and in an even (or zero) number $j$ of $k$-simplexes of $C_2$. Say that $n$ $k$-simplexes containing $d$ are in both $C_1$ and $C_2$. Then $d$ is in $(i - n) + (j - n)$ number of $k$-simplexes of $C_1 + C_2$, since every $k$-simplex in both $C_1$ and $C_2$ is not in the sum. However, notice that $i + j$ is odd, and $2n$ is even. Therefore, $(i - n) + (j - n) = i + j - 2n$ is odd, so $d$ is in an odd number of $k$-simplexes of $C_1 + C_2$, and therefore $d \in \partial(C_1 + C_2)$.

Therefore, $\partial(C_1 + C_2) = \partial(C_1) + \partial(C_2)$. □

We call a $k$-chain a $k$-cycle if it does not have a non-empty boundary. That is

**Definition 17. Cycle** *Let $\mathfrak{G}$ be a grating, and let $C$ be a k-chain on $\mathfrak{G}$. $C$ is a cycle if $\partial(C) = \emptyset$.*

From our discussion of boundaries of 0-chains, it is clear that all 0-chains are 0-cycles. Although it may not be immediately apparent, all boundaries are cycles. For a picture of why this is true, consider Figure 11 again. The boundary of the boundary in that figure are all the vertices that are in an odd number of the darkened edges. By inspection, we see that every vertex of one edge in the boundary is in either one or three other edges. Therefore, no vertices are in an odd number of edges, and the boundary of the boundary is the empty set, making the boundary a cycle. Here is the general proof.

**Theorem 17.** *All boundaries are cycles.*

*Proof.* Let $\mathfrak{G}$ be a grating, and a $k$-chain $C$ $(k = 0, 1)$ be a boundary in $\mathfrak{G}$. That is, there exists a $(k+1)$-chain $T$ such that $\partial(T) = C$. We wish to show that $\partial(C) = \emptyset$. There are two cases of interest to us, either $T$ is a 2-chain, or $T$ is a 1-chain. The latter case can be dispensed with easily, since this implies that $C$ is a 0-chain, and all 0-chains are cycles.

Now let us consider the case that $T$ is a 2-chain, so $C$ is a 1-chain. We proceed by induction on the number $k$ of 2-simplexes in $T$. Let $k = 1$, so $T$ is a single face. Then $\partial(T) = C$ is the four edges of the rectangular face. Each vertex of the rectangle is clearly in exactly two edges, so $\partial(C) = \emptyset$. This proves the base base. Now, assume that, for some integer $k$, any 2-chain with $k$ 2-simplexes has a boundary which is a cycle. Now let $T$ be a 2-chain composed of $k+1$ 2-simplexes. We can split $T$ into $S$ and $K$, where $S$ is a 2-chain with $k$ faces, and $K$ is a 2-chain consisting of the single other face of $T$.

We see that the boundary $C$ of $T$ is

$$C = \partial(T) = \partial(S + K) = \partial(S) + \partial(K)$$

by the additivity of the boundary operator. Since $S$ has $k$ 2-simplexes, $\partial(\partial(S)) = \emptyset$, and we have already seen the a single face has a cycle for its boundary. Therefore,

$$\partial(C) = \partial(\partial(S)) + \partial(\partial(K)) = \emptyset.$$

Therefore, for 2-chains of $k+1$ 2-simplexes, the boundary is a cycle. By induction, this is true for all 2-simplexes. Therefore, all boundaries are cycles. □

We now introduce the important idea of homology. Let us first define the term, then see what it means

**Definition 18. Homologous** *Let $\mathfrak{G}$ be a grating. Two $k$-chains $C_1$ and $C_2$ in $\mathfrak{G}$ are homologous, denoted $C_1 \sim C_2$ if $C_1 + C_2$ is a boundary for a $(k+1)$-chain.*

Clearly, for our purposes, homology will not be very useful for 2-chains, since we do not define a boundary of a 3-chain, as mentioned above. However, for $k = 0, 1$, this idea will provide the key to the Jordan Curve Theorem. The best way to see what homology means is to look at some examples. In Figure 12, we see

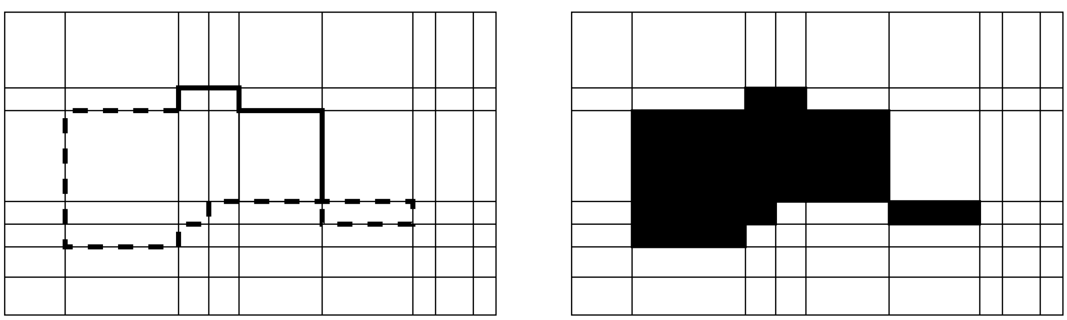

Figure 12: Two Homologous 1-Chains

two 1-chains, one denoted by a solid line, the other by a dotted line. Neither is a boundary for a 2-chain. However, we see that, adding the two together, the result is a boundary for the 2-chain on the right of Figure 12. Another way to see what homologous means for 1-chains is to consider what Theorem 17 tells us about boundaries. Since all boundaries are bounded by the empty set, two 1-chains are homologous if, in adding them, their boundaries "cancel out." We see this in Figure 12, both 1-chains begin and end in the same place. Therefore, when performing chain addition on their boundaries, the results is $\emptyset$. Homology turns out to be an equivalence relation on a grating:

**Theorem 18.** *Let $\mathfrak{H}$ be a grating, then the homology relation $\sim$ defines an equivalence relation on $\mathfrak{H}$.*

*Proof.* To prove an equivalent relation, we must prove that $\sim$ is all of the following: reflexive, symmetric, transitive, and additive.

- Let $\mathfrak{H}$ be a grating, and let $C$ be a $k$-chain. We wish to show that $C \sim C$. Note that $C + C = \emptyset$. Since the empty set is a boundary (of the empty set), $C \sim C$.

- Let $C_1$ and $C_2$ be $k$-chains. We wish to show that if $C_1 \sim C_2$, $C_2 \sim C_1$. Since $C_1 \sim C_2$, $C_1 + C_2$ is the boundary for some $(k+1)$-chain $T$. That is $\partial(T) = C_1 + C_2$. By the commutativity of the $k$-chain group, $\partial(T) = C_1 + C_2 = C_2 + C_1$, so $C_2 \sim C_1$.

- Let $C_1, C_2$ and $C_3$ be $k$-chains. We wish to prove transitivity, so assume that $C_1 \sim C_2$ and $C_2 \sim C_3$. We wish to show that $C_1 \sim C_3$. Since $C_1 \sim C_2$, there exists a $(k+1)$-chain $T_1$ such that $\partial(T_1) = C_1 + C_2$. Likewise, there exists $T_2$ such that $\partial(T_2) = C_2 + C_3$. Now, by Theorem 16,

$$\begin{aligned}
\partial(T_1 + T_2) &= \partial(T_1) + \partial(T_2) \\
&= (C_1 + C_2) + (C_2 + C_3) \\
&= C_1 + (C_2 + C_2) + C_3 \\
&= C_1 + C_3.
\end{aligned}$$

  Therefore, $C_1 + C_3$ is the boundary of the $(k+1)$-chain $T_1 + T_2$, so $C_1 \sim C_3$[19]

- Finally, we wish to show that, if $C_1 \sim C_2$ and $C_3 \sim C_4$, $C_1 + C_3 \sim C_2 + C_4$ for $k$-chains $C_1, C_2, C_3$, and $C_4$. Again, there exists $(k+1)$-chains $T_1$ and $T_2$ such that $\partial(T_1) = C_1 + C_2$ and $\partial(T_2) = C_3 + C_4$. Therefore, using the group properties of chain addition,

$$\begin{aligned}
\partial(T_1 + T_2) &= \partial(T_1) + \partial(T_2) \\
&= (C_1 + C_2) + (C_3 + C_4) \\
&= C_1 + (C_2 + (C_3 + C_4)) \\
&= C_1 + ((C_2 + C_3) + C_4) \\
&= C_1 + ((C_3 + C_2) + C_4) \\
&= (C_1 + (C_3 + C_2)) + C_4 \\
&= ((C_1 + C_3) + C_2) + C_4 \\
&= (C_1 + C_3) + (C_2 + C_4)
\end{aligned}$$

  Therefore, $C_1 + C_3$ and $C_2 + C_4$ form a boundary of $T_1 + T_2$, so $C_1 + C_3 \sim C_2 + C_4$.

[19] [2] p.90

Thus $\sim$ is an equivalence relation. □

At this point, we have all the necessary basic tools to prove the Jordan Curve Theorem. We shall now move to establish the last theorems upon which the Jordan Curve Theorem relies. These draw upon the ideas that have been established in these last two sections.

## 4 Jordan Curve Theorem

This section follows closely the ideas and proof techniques of § 16-18 in Henle. Sections 2 and 3 of this paper should have been relatively simple for readers with a background in abstract algebra or real analysis, even if many of the ideas contained therein are new. This section is most likely unfamiliar territory for most, and the proofs are somewhat tricky, so a careful reading is urged.

The first theorem we wish to prove is the Fundamental Lemma, which states

**Theorem 19. (Fundamental Lemma)** *Every 1-cycle is a boundary of exactly two complementary 2-chains in the plane.*

Basically, this is the Jordan Curve Theorem for gratings. Since it refers to 1-cycles instead of Jordan Curves, it allows for the possibility that the cycle will cross itself but other than that, it states that a closed loop divides the set it is drawn on into two distinct sections, exactly as the Jordan Curve Theorem does. An example can be seen in Figure 13. As you can clearly see, the curve divides the grating into two 2-chains, even though the 1-cycle in this case crosses over itself. The proof of this lemma follows:

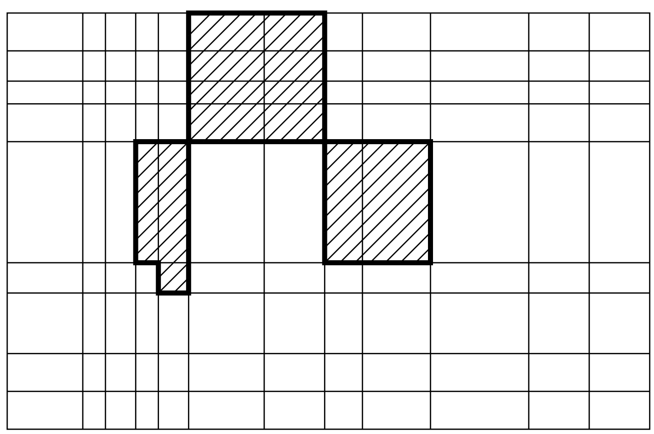

Figure 13: Fundamental Lemma Example

*Proof.* Let $\mathfrak{G}$ be a grating, and $\lambda$ be a 1-cycle on $\mathfrak{G}$. We will prove this theorem in two parts. First, we shall prove that $\lambda$ is the boundary of at least two 2-chains on the grating, and then we prove that it is the boundary of at most two.

- We prove that there are at least two 2-chains with boundary $\lambda$. We do so by induction on the number of lines on the grating. When there are 4 lines on $\mathfrak{G}$, we are dealing with an empty rectangle, so there are only two 1-cycles: the rectangle and the empty set, and two 2-chains, the rectangle face, and the rest of the plane not in the grating. The rectangle edge is the boundary for the rectangle face and the rest of the plane. The empty set is the boundary for the empty set and the entire plane. Additionally, these are complementary 2-chains. Therefore, the Fundamental Lemma is true for when the number of lines on the grating is four.

  Now, let us assume that the claim "every 1-cycle is the boundary for at least two complementary 2-chains" is true for some grating $\mathfrak{G}$. We can define a new grating $\mathfrak{G}^+$ be adding in a line $\ell$ to $\mathfrak{G}$. Without loss of generality, we can assume this line is horizontal. Let $\lambda$ be a 1-cycle on $\mathfrak{G}^+$. We would like to remove and edges of $\lambda$ on $\ell$, since then we would be dealing with a 1-cycle on $\mathfrak{G}$. We do so by using chain addition. Let $K$ be the chain of 2-simplexes in $\mathfrak{G}^+$ whose lower edges rest on $\lambda$ and $\ell$. That is, $K$ is all the faces of our new grating whose bottom edges are segments of both $\lambda$ and $\ell$. Therefore, $\partial(K)$ contains the edges that we wish to remove from $\lambda$. Let $\mu = \lambda + \partial(K)$. By construction, $\mu$ has no edges on $\ell$, since all edges of $\lambda$ that were on $\ell$ where removed by adding $\partial(K)$. Notice that

  $$\partial(\mu) = \partial(\lambda) + \partial(\partial(K)),$$

  however, the boundary of a cycle is the empty set, so $\partial(\mu) = \emptyset$, and $\mu$ is a 1-cycle.

  We have now constructed $\mu$ as a 1-cycle on $\mathfrak{G}$, so, by hypothesis, $\mu$ is the boundary for at least two complementary 2-chains, $S_1$ and $S_2$. Now we have to add back in the 2-simplexes that $\lambda$ was the boundary of, which we removed in order to get $\mu$. Those 2-simplexes were all contained in $K$. So let $T_1 = S_1 + K$, and $T_2 = S_2 + K$. We see that

  $$\partial(T_1) = \partial(S_1 + K) = \partial(S_1) + \partial(K) = \mu + \partial(K) = \lambda,$$

  and similarly $\partial(T_2) = \lambda$. Since $S_1$ and $S_2$ were distinct, $T_1$ and $T_2$ are.

  We show that they are complementary by noting that every 2-simplex is either in $S_1$ or $S_2$, but not both, since $S_1$ and $S_2$ are complementary by the induction hypothesis. The only 2-simplexes that differ from $S_1$ to $T_1$ and $S_2$ to $T_2$ are the ones in $K$. Let $k \in K$ be a 2-simplex. Assume $k \in S_1$. Then $k \notin T_1$ and, since $k \in S_1$, $k \notin S_2$, $k \in T_2$. A parallel argument holds if $k \in S_2$. Therefore, $T_1$ and $T_2$ are complementary.

  Since any grating can be built out of an empty rectangle by adding one line at a time, by induction, on any grating $\mathfrak{H}$, every 1-cycle $\lambda$ is the boundary for at least two 2-chains.

Before continuing on to the second part, let us take a moment here to get a visual idea of what is going on in the first part of the proof. The base case of the induction argument should be clear enough. We next assume that the statement "any 1-cycle divides the plane into at least two 2-chains" is true for some grating $\mathfrak{G}$.

Now, we add one line $\ell$ to the grating $\mathfrak{G}$, call the new grating $\mathfrak{G}^+$ and pick some 1-cycle on $\mathfrak{G}^+$ (for simplicity, we assume the line is horizontal, but since we can reorient the grating however we want, this does not limit the proof). The 1-cycle either has 1-simplexes lying on the new line, or it does not. If it does not, then the 1-cycle $\lambda$ is on the old grating $\mathfrak{G}$. If it does share edges with the new line, as it does in Figure 14 (a), then we make a new 1-cycle by chain addition of $\lambda$ and the boundary of the faces that sit "on top of" $\ell$ (we call this 2-chain $K$). In our example, the new 1-cycle $\mu$ is shown as a solid line, while the boundary of the $K$ is shown as dotted. Since $\mu$ is on the old grating $\mathfrak{G}$, we know that it is the boundary for at least two 2-chains. We then add in the boundary of $K$ to get back to $\lambda$ and our new grating. As we show in the formal argument, since $\mu$ is the boundary of at least two 2-chains, the same is true for $\lambda$.

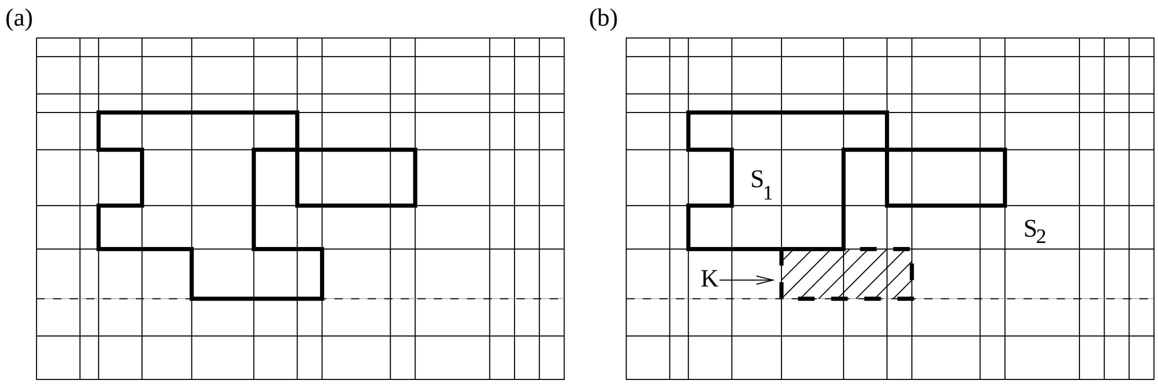


Figure 14: $\mathfrak{G}^+$ and 1-cycle

- We now prove that every 1-cycle divides the plane into at most two 2-chains. Let $\mathfrak{G}$ be a grating, and let $\lambda$ be a 1-cycle on that grating. From the first part of this proof, we know that there are at least two complementary 2-chains that $\lambda$ forms the boundary of. Let us call the two that we know exist $T_1$ and $T_2$, so $\lambda = \partial(T_1) = \partial(T_2)$ and $T_1^c = T_2$. Now, let us pick another 2-chain $S$ with the boundary $\lambda$. We wish to show that $S$ is equal to either $T_1$ or $T_2$. We see that

$$\partial(T_1 + S) = \partial(T_1) + \partial(S) = \lambda + \lambda = \emptyset.$$

Therefore, $T_1 + S$ is a 2-cycle.

We now prove that the only 2-cycles are $\emptyset$ and the whole plane. Clearly, $\partial(\emptyset) = \emptyset$, so the empty set is a 2-cycle. Every edge in a grating is in two 2-simplexes. Therefore, every edge in the plane is contained in an even number of 2-simplexes, so the boundary of the plane is again the empty set. If some 2-chain other than the plane or the empty set was a 2-cycle, than it must contain at least one 2-simplex. However, if it is not the entire plane, it cannot contain every pair of 2-simplexes that contain a given edge. Therefore, its boundary cannot be $\emptyset$.

Thus, $T_1 + S$ must either be the empty set or the entire plane. If the former is true, $T_1 = S$. If the latter is true, $T_1^c = S$. Therefore, $S$ is not a distinct 2-chain bounded by $\lambda$, so $\lambda$ is the boundary for at most two complementary 2-chains. This proves the Fundamental Lemma.[20]

□

Using the Fundamental Lemma, we can prove some quick corollaries. The first is that if an open set has at most one "hole" in it (for our purposes a hole in the plane can be represented by a connected closed set), every 1-cycle in the open set is still a boundary for some 2-chain in the set. Pictorially, this is pretty clear, as in Figure 15. In order for the 1-cycle not to be the boundary for some 2-chain, both 2-chains in the unbroken plane that the Fundamental Lemma tells us the 1-cycle is the boundary of would have to intersect the closed set. However, the only way for this to happen is if the 1-cycle itself intersects the closed set, contradicting our initial assumption.

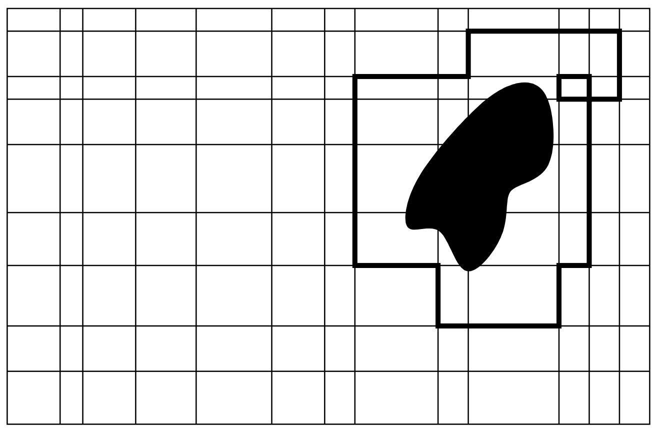

Figure 15: Example for Corollary I

**Theorem 20. (Corollary I)** *Let $F$ be a connected closed set, and $G$ the complimentary open set. Let $\mathfrak{G}$ be a grating. Then every 1-cycle in $G$ on $\mathfrak{G}$ is the boundary of a 2-chain in $G$.*

*Proof.* Let $\mathfrak{G}$ be a grating and $\lambda$ be a 1-cycle on $\mathfrak{G}$. By the Fundamental Lemma, there exists complementary 2-chains $S_1$ and $S_2$ such that $\partial(S_1) = \partial(S_2) = \lambda$. Proceed by contradiction. Assume that both $S_1$ and $S_2$ intersect the connected closed set $F$. Then, since $S_1$ and $S_2$ exhaust the plane, $S_1 \cap F$ and $S_2 \cap F$ are nonempty and partition $F$. Since $F$ is connected, by Definition 12, one of $S_1 \cap F$ and $S_2 \cap F$ contains a point near the other. However, this means that $F$ contains a point of the boundary of $S_1$ and $S_2$. The boundary of these two sets is $\lambda$, so $F$ contains a point of $\lambda$. This is a contradiction, so the corollary is proven.[21] □

Once we have shown that an open set with one "hole" still has at least one boundary for every 1-cycle, the next question is what happens in open sets with two holes? Thinking of examples (one of which is in

[20] [2] p.91
[21] [2] p.94

Figure 16), we see that it is possible for a 1-cycle to not be a boundary for a 2-chain (the 1-cycle denoted by a solid line in the figure), however this is not necessarily true (the dotted line is a boundary for the shaded 2-chain in the figure). Our second corollary of the Fundamental Lemma sums up this line of reasoning.

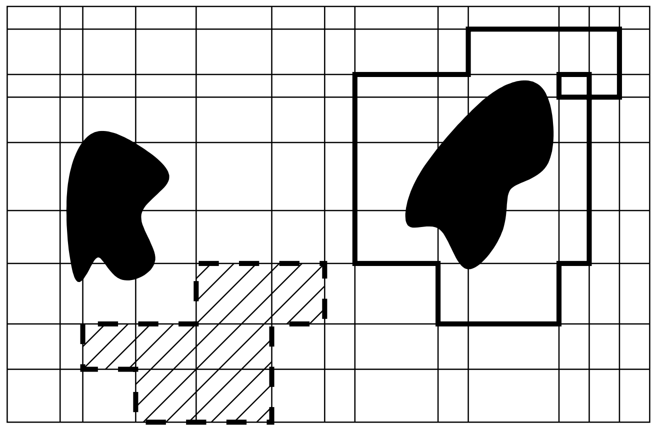

Figure 16: Example for Corollary II

**Theorem 21. (Corollary II)** *Let $F$ be a closed set consisting of two connected components, and let $G$ be the complementary open set. Let $\mathfrak{G}$ be a grating. Then $\mathfrak{G}$ may contain 1-cycles that are not boundaries in $G$. However, any two such 1-cycles are homologous in $G$.*

*Proof.* The first part of the corollary is simple, since we need only show some configuration of $G$, $F$, and a 1-cycle that is not a boundary in $G$. Again, referring to Figure 16, where the blacked out sets are $F$, we see that the 1-cycle denoted by solid lines is not a boundary in $G$, since the interior 2-chain is intersected with one part of $F$, and the exterior 2-chain also intersects $F$.

The non-trivial part of this proof therefore, is proving the homologous nature of those 1-cycles. Let $F$ be a closed set consisting of two disjoint connected closed sets, $F_1$ and $F_2$. Let $G$ be the open compliment of $F$, and $\mathfrak{H}$ be a grating. We have shown that there may exist 1-cycles that are not boundaries in $G$. Let $\lambda$ and $\mu$ on $\mathfrak{H}$ be two such 1-cycles.

By the Fundamental Lemma, we know that $\lambda$ is the boundary for 2-chains $T_1$ and $T_2$ on $\mathfrak{H}$. By Corollary I, ignoring $F_2$, so $F_1$ is the only closed set in the plane, we see that $\lambda$ must still be the boundary for either $T_1$ or $T_2$. Whichever one $\lambda$ is the boundary for, this meaning that $F_1$ must be contained in the other 2-chain ($T_1$ if $\lambda$ is the boundary for $T_1$, or vica versa). A similar argument follows for $F_2$. Since $\lambda$ is not a boundary in $G$, we know that $F_1$ and $F_2$ cannot both be in either $T_1$ or $T_2$. We can therefore assume, without loss of generality, that $F_1$ is in $T_1$ and $F_2$ is in $T_2$. By an identical argument, we know that, disregarding $F$, $\mu$ is the boundary for $S_1$ and $S_2$, and that $F_1$ is in $S_1$ and $F_2$ is in $S_2$ (again, without loss of generality).

Now, we want to show that $\lambda \sim \mu$ in $G$. Therefore we need to show that $\lambda + \mu$ is a boundary in $G$. That means we must find a 2-chain that avoids both $F_1$ and $F_2$. We have $S_1$ and $T_1$, both of which are disjoint from $F_2$. Therefore, $T_1 + S_1$ is also disjoint from $F_2$. Now, if a 2-simplex of $T_1$ intersects $F_1$, than since $F_1$ is also contained in $S_1$, that 2-simplex must also be contained in $S_1$. Therefore, no face intersecting $F_1$ is in $T_1 + S_1$, so $F_1$ is disjoint from that sum. We also see that

$$\partial(T_1 + S_1) = \partial(T_1) + \partial(S_1) = \lambda + \mu.$$

Therefore, $\lambda \sim \mu$.[22] □

So far we have been dealing only with the properties of $k = 1$ homology, that is, homology of sets of edges. We will take a quick moment to prove a useful property of homology for 0-chains. In general, we have seen that two 1-chains are homologous if there is a way to bend one into the other while remaining inside the set. Similarly, the conclusion that we will reach for 0-simplexes is that they are homologous if we can push one into the other while remaining in the set.

**Theorem 22.** *Let $G$ be an open set, and let $a, b \in G$. Then $a \sim b$ in $G$ iff $a$ and $b$ can be connected by a polygonal chain in $G$.*

*Proof.*

($\Longrightarrow$) Let $G$ be an open set, $a, b \in G$ such that $a \sim b$, and $\mathfrak{G}$ be a grating such that $a$ and $b$ are located at vertices of $\mathfrak{G}$. $a$ and $b$ are vertices of the grating, hence they are 0-simplexes. Therefore, $a \sim b$ tells us that there exists a 1-chain $\lambda$ such that $\partial(\lambda) = a + b$. The 1-chain $\lambda$ must at least consist of a polygonal chain connecting $a$ and $b$, since the endpoints of $\lambda$ are $a$ and $b$. Therefore, $a$ and $b$ are connected by a polygonal chain in $G$.

($\Longleftarrow$) Let $G$ be an open set, and $a, b \in G$, and $a$ and $b$ connected by a polygonal chain in $G$. Then we can construct a grating $\mathfrak{H}$ in the following way. First, place a rectangle in $G$ that encloses $a$ and $b$ with edges parallel to the axes. Then, place lines on the rectangle that overlay each segment of the polygonal chain. Since the chain consists of a finite number of straight lines parallel to the axes, we can do this. Therefore, the polygonal chain is now a 1-chain on $\mathfrak{H}$, with boundary $a + b$. So $a \sim b$.

□

The previous theorem is very similar to the Polygonal Chain Theorem. Its usefulness is that it allows us to connect the idea of connectedness (which is linked to polygonal chains by Theorem 13) to the idea of homology.

We are now almost to point of being able to prove the Jordan Curve Theorem. What we are missing is a way to determine whether there is, in a plane with two holes in it, a way to draw a polygonal chain between to points that "misses" the holes. The theorem that tells us the conditions that allow this is Alexander's Lemma. Again, this and following proofs follow closely the treatment of Henle (§17 and §18).

[22] [2] p.94

**Theorem 23. (Alexander's Lemma)** *Let $F_1$ and $F_2$ be compact closed sets, with complementary open sets $G_1$ and $G_2$ respectively. Let $a, b$ be points in the plane such that $a, b \in G_1 \cap G_2$ and $a \sim b$ in both $G_1$ and $G_2$. Let $\lambda_1$ and $\lambda_2$ be chains in $G_1$ and $G_2$ respectively such that $\partial(\lambda_1) = \partial(\lambda_2) = a + b$. If $\lambda_1 \sim \lambda_2$ in $G_1 \cup G_2$, then $a \sim b$ in $G_1 \cap G_2$.*

Let us take a moment to break down what this theorem is saying. In Figure 17, we see one possible configuration with which the lemma deals. In this case, we see that $\lambda_1 + \lambda_2$ (the loop created by the two chains) is not a boundary in the union of the complements of $F_1$ and $F_2$ ($F_1^c \cup F_2^c$ corresponds to the intersection of $F_1$ and $F_2$). The problem is that, in order to avoid going through $F_2$, $\lambda_2$ goes around the intersection of $F_1$ and $F_2$ in one direction, and $\lambda_1$ is constrained to go the other. Sure enough, we see that, considering the intersection of the complements of $F_1$ and $F_2$ (that is $F_1 \cup F_2$), we cannot draw a polygonal chain between $a$ and $b$. From Theorem 22, we know that this is equivalent to saying that $a \not\sim b$.

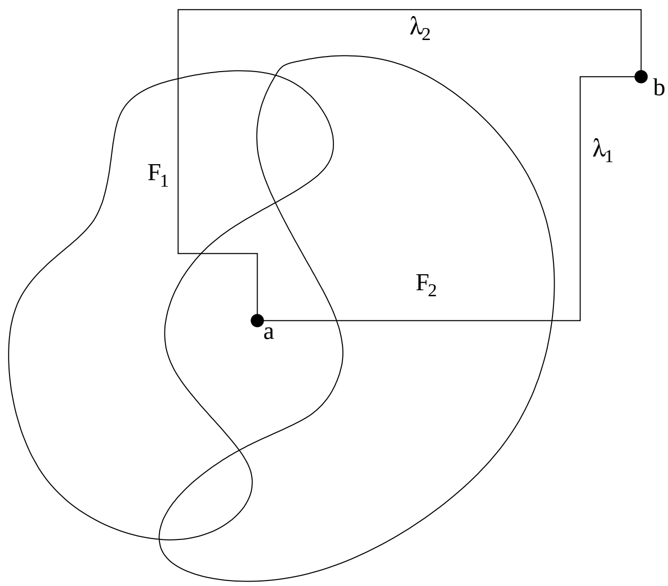


Figure 17: Example of Alexander's Lemma

An example where $a \sim b$ is shown in Figure 18. In this case, there is a space between $F_1$ and $F_2$, allowing us to avoid encircling the intersection of the two closed sets. Therefore, $\lambda_1 \sim \lambda_2$ when we intersect $F_1$ and $F_2$, since we can construct a grating that has a 2-chain of which $\lambda_1 + \lambda_2$ is a boundary. We also see that we can draw a polygonal path between $a$ and $b$ that avoids both $F_1$ and $F_2$. $\lambda_1$ is an example of just such a path. Clearly then, a proof of Alexander's Lemma will rely upon some idea of there being a gap between the two closed sets. In order to exploit this idea, we first need another lemma.

**Lemma 24.** *Let $F_1$ and $F_2$ be disjoint compact sets. Then there exists a constant $\epsilon > 0$ such that if $a \in F_1$ and $b \in F_2$, then $||a - b|| > \epsilon$.*

*Proof.* Proceed by contradiction. Suppose that, for all $\epsilon > 0$, there exists $\alpha \in F_1$ and $\beta \in F_2$ such that

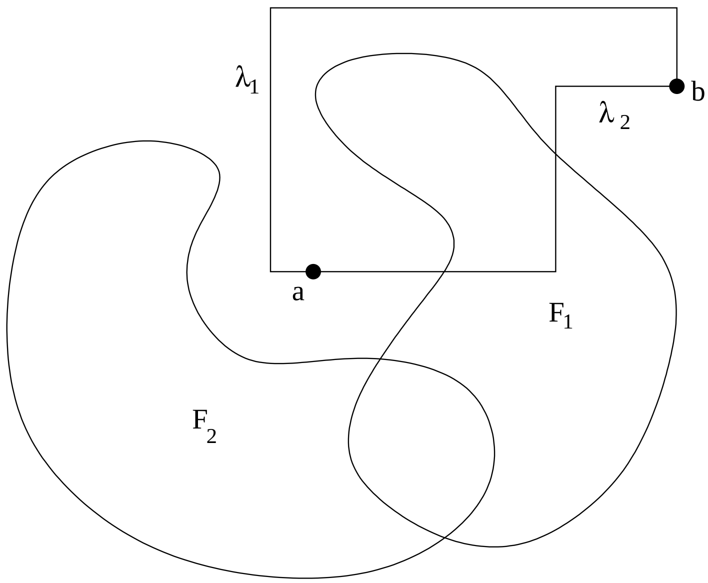


Figure 18: Example of Alexander's Lemma

$||\alpha - \beta|| < \epsilon$. Then we can construct a sequence $\{a_i\} \in F_1$ and $\{b_i\} \in F_2$ such that $||a_i - b_i|| < \frac{1}{i}$. Since $F_1$ is compact, $\{a_i\}$ has a near point $a$ in $F_1$. Since $||a_i - b_i|| < \frac{1}{i}$, the near point of $\{b_i\}$ must be the same as the near point for $\{a_i\}$. Since $F_2$ is closed, it contains all its near points. Therefore $a \in F_1$ and $a \in F_2$. This contradicts our claim that $F_1$ and $F_2$ are compact.[23] □

We can now prove Alexander's Lemma.

*Proof.* Let $F_1$ and $F_2$ be compact closed sets with complementary open sets $G_1$ and $G_2$. Let $a, b \in G_1 \cap G_2$, $\mathfrak{G}$ be a grating such that $a \sim b$, and let $\lambda_1, \lambda_2$ be 1-chains such that $\partial(\lambda_1) = \partial(\lambda_1) = a + b$. Finally, let $\lambda_1 \sim \lambda_2$ in $G_1 \cup G_2$.

Since $\lambda_1 \sim \lambda_2$, there exists a 2-chain $S$ in $G_1 \cup G_2$ such that $\partial(S) = \lambda_1 + \lambda_2$. Consider $S \cap F_1$ and $S \cap F_2$. We claim that these two subsets are disjoint. Assume that they are not. Then there exists some 2-simplex that is in both $S \cap F_2$ and $S \cap F_1$. Therefore, this 2-simplex is in both $F_1$ and $F_2$, so it cannot be in $G_1 \cup G_2$. This implies that $S$ is not in $G_1 \cup G_2$, which is the desired contradiction.

By the previous lemma, we know that there exists $\epsilon > 0$ such that every pair of points $a, b$, where $a \in S \cap F_1$ and $b \in S \cap F_2$, are separated by at least $\epsilon$. That is $||a - b|| > \epsilon$. We therefore can subdivide our grating $\mathfrak{G}$ such that no face intersects both $S \cap F_1$ and $S \cap F_2$. We do so by adding lines to $\mathfrak{G}$ until the separation between each pair of lines in the grating is less than $\epsilon/2$. This insures that no face can "span the gap" between the two sets. Call this new grating $\mathfrak{G}^+$.

Now consider the 2-chain on our new grating of faces of $S$ intersecting $F_2$. Call this chain $T$. By construction, $T$ is disjoint from $F_1$, since $S \cap F_1$ does not intersect $S \cap F_2$ on $\mathfrak{G}^+$. Define $\lambda$ to be $\lambda_1 + \partial(T)$.

[23][2] p.98

What this does is "guide" our chain around $F_2$, without intersecting $F_1$. We see that

$$\partial(\lambda) = \partial(\lambda_1 + T) = \partial(\lambda_1) + \partial(\partial(T)) = \partial(\lambda_1) = a + b.$$

We now just need to prove that $\lambda$ is disjoint from both $F_1$ and $F_2$. Since $\lambda_1$ and $T$ are disjoint from $F_1$, so is $\lambda$. Now $S + T$ is disjoint from $F_2$, so

$$\partial(S + T) = \partial(S) + \partial(T) = \partial(S) + \lambda_1 + \lambda_2 = \lambda + \lambda_2.$$

Therefore, adding $\lambda_2$ to each side, $\partial(S + T) + \lambda_2 = \lambda$. Since $\lambda_2$, $S$ and $T$ are all disjoint from $F_2$, so is $\lambda$. Therefore, $\partial(\lambda) = a + b$, and $\lambda \in G_1 \cap G_2$. It follows that $a \sim b$ in $G_1 \cap G_2$.[24] □

We now come to the last theorem before the Jordan Curve Theorem itself, the Jordan Curve Theorem for Paths. This theorem states that a path does not divide the plane. Just as with the Jordan Curve Theorem, this is a very obvious fact: a finite path cannot divide the plane into two disconnected pieces, you can always find a way to go around. This proof uses the method of bisection first introduced in the proof for Theorem 11.

**Theorem 25. (Jordan Curve Theorem for Paths)** *Paths do not divide the plane.*

*Proof.* Let $\gamma$ be a path. We proceed by contradiction. Let $a$ and $b$ be points in the plane on $\gamma$ such that $a$ and $b$ cannot be connected in $\gamma^c$. Since $\gamma$ is a path, there exists a topological transformation from $[0, 1]$ to $\gamma$. We can divide $\gamma$ in half by considering $\gamma_1 = T[0, 1/2]$ and $\gamma_2 = T[1/2, 1]$. We will show that, if $\gamma$ divides $a$ from $b$, exactly one of the halves of $\gamma$ must do so as well.

At this point we do something very tricky: proof by contradiction within a proof by contradiction. That is, we assume that neither of $\gamma_1$ or $\gamma_2$ divides $a$ from $b$ and show that this implies that $\gamma$ does not divide them either. This may seem to some suspiciously like assuming what we wish to prove, but since we are taking the result and using it in our original proof by contradiction, everything works out in the end. Therefore, assuming the neither $\gamma_1$ or $\gamma_2$ divides $a$ from $b$, then there exists a polygonal path $\lambda_1$ connecting $a$ and $b$ that does not intersect $\gamma_1$, and another polygonal path $\lambda_2$ that does not intersect $\gamma_2$. Since $\gamma_1$ and $\gamma_2$ are paths, they are closed, therefore $\gamma_1^c$ and $\gamma_2^c$ are open.

We see that $\gamma_1^c \cup \gamma_2^c$ is the entire plane minus the center point of $\gamma$. Note that since $\partial(\lambda_1) = \partial(\lambda_2) = a + b$, $\partial(\lambda_1 + \lambda_2) = \emptyset$, so $\lambda_1 + \lambda_2$ is a 1-cycle. By Corollary I, $\lambda_1 + \lambda_2$ is a boundary in $\gamma_1^c \cup \gamma_2^c$, so $\lambda_1 \sim \lambda_2$ in $\gamma_1^c \cup \gamma_2^c$. By Alexander's Lemma, this means that $a \sim b$ on $\gamma_1^c \cap \gamma_2^c$. However, $\gamma_1^c \cap \gamma_2^c = \gamma_1 \cup \gamma_2 = \gamma$, and if $a \sim b$, then $a + b$ is a boundary for a polygonal chain, contradicting our assumption that $\gamma$ divides $a$ from $b$.

With that, we now see that if $\gamma$ divides $a$ from $b$, than either $\gamma_1$ or $\gamma_2$ divides $a$ from $b$. Assume without loss of generality that it is $\gamma_1$. We can now divide $\gamma_1$ exactly as we did $\gamma$. By an argument identical to the one used for $\gamma$, we see that exactly one of these halves must divide $a$ from $b$. We can continue doing this, generating a sequence of paths, all shorter than the previous. The limit of this sequence is a single point.

[24] [2] p.98

Therefore, if $\gamma$ divides $a$ from $b$, than there exists a point $p$ such that $p$ divides $a$ from $b$. However, we can clearly see that, for any points $a, b$, and $p$, we can find a polygonal path that has endpoints in $a$ and $b$ but does not intersect $p$. This contracts our assumption that $\gamma$ divides $a$ from $b$, therefore, paths cannot divide the plane. □

Now, at long last, we are able to prove the Jordan Curve Theorem. Looking back at the statement of Theorem 1, we see that we can split it into four statements. First, that any Jordan curve divides the plane into at least two pieces, then that it divides the plane into at most two pieces. Next we need to prove that the Jordan curve is the boundary for both pieces, and lastly that one piece is bounded, and the other unbounded. Remember that something can be unbounded (not able to be contained in a rectangle) yet still have a boundary. With this general scheme in mind, restate the Theorem and then proceed with the proof.

**Jordan Curve Theorem.** *Let $\mathfrak{J}$ be a Jordan curve. Then the compliment of $\mathfrak{J}$ in the plane, $\mathfrak{J}'$, is not connected but consists of two disjoint connected pieces, one of which is bounded and one of which is not bounded. The curve $\mathfrak{J}$ forms the boundary for both pieces.*[25]

*Proof.* Let $\mathfrak{J}$ be a Jordan curve. This proof was based on the techniques laid out in Henle.[26]

- **$\mathfrak{J}$ divides the plane into at least two pieces.**

  To prove this, we must produce two points in the complement of $\mathfrak{J}$ that cannot be connected in the complement. First, we wish to divide $\mathfrak{J}$ into two paths. We do so by selecting point $a$ and $b$ on $\mathfrak{J}$. These points define two paths, $\gamma_1$ and $\gamma_2$ on $\mathfrak{J}$ such that $\gamma_1 \cup \gamma_2 = \mathfrak{J}$ and $\gamma_1 \cap \gamma_2 = \{a, b\}$. Since $a \neq b$, there exists $\epsilon > 0$ such that $d(a, b) > \epsilon$. Therefore, we can construct a square $\sigma$ with side length $\epsilon/2$ centered at $a$ that does not contain $b$. For reasons that will be clear later, we wish that no edge of $\sigma$ intersects both $\gamma_1$ and $\gamma_2$. We insure that this is true by using Lemma 24. That is, at any given distance $\delta$, the lemma tells us that the sections of $\gamma_1$ and $\gamma_2$ outside the ball of radius $\delta$ about $a$ must be separated by some $\eta$. Though we will not prove this, it should be clear that we can therefore pick some side length less than our original $\epsilon/2$ that is small enough not to intersect both $\gamma_1$ and $\gamma_2$

  We have shown that the Jordan Curve Theorem holds for rectangles, therefore, since $a \in \sigma$, $b \notin \sigma$, and $\gamma_1$ and $\gamma_2$ connect $a$ and $b$, $\gamma_1$ and $\gamma_2$ intersect $\sigma$. Our goal now is to show that $\sigma$ contains points on "either side" of $\mathfrak{J}$.

  We can select a grating $\mathfrak{G}$ on which the edges of $\sigma$ form a 1-chain. Let $\lambda$ be the 1-chain of edges of $\sigma$ intersecting $\gamma_1$. The chain $\lambda$ is not empty, and by our unproven assumption, since $\gamma_1$ and $\gamma_2$ do not intersect the same edges of $\sigma$, $\lambda \neq \sigma$. Consider $\partial(\lambda)$. $\lambda$ must consist of at least one edge of $\mathfrak{G}$, and possibly more. For an example, see Figure 19. In that case, the highlighted sections of the grating are the elements of $\lambda$ ($\sigma$ has been chosen as the outside edges of $\mathfrak{G}$ for simplicity). It is therefore clear that

[25] [2] p.81
[26] [2] p.99

$\partial(\lambda)$ consists of a finite number of pairs of elements, since every polygonal chain has a boundary of exactly two 0-simplexes. Call these element pairs $p_1, q_1, p_2, q_2, \cdots, p_n, q_n$. Now, none of these elements can be in $\mathfrak{J}$. If one was, than it would be in two 1-simplexes of $\sigma$ intersecting $\mathfrak{J}$, and therefore not in the boundary. It is our claim that at least one of these pairs of points $p_i, q_i$ is separated by $\mathfrak{J}$. That is, by the Polygonal Chain Theorem, a polygonal chain from $p_i$ to $q_i$ intersects $\mathfrak{J}$.

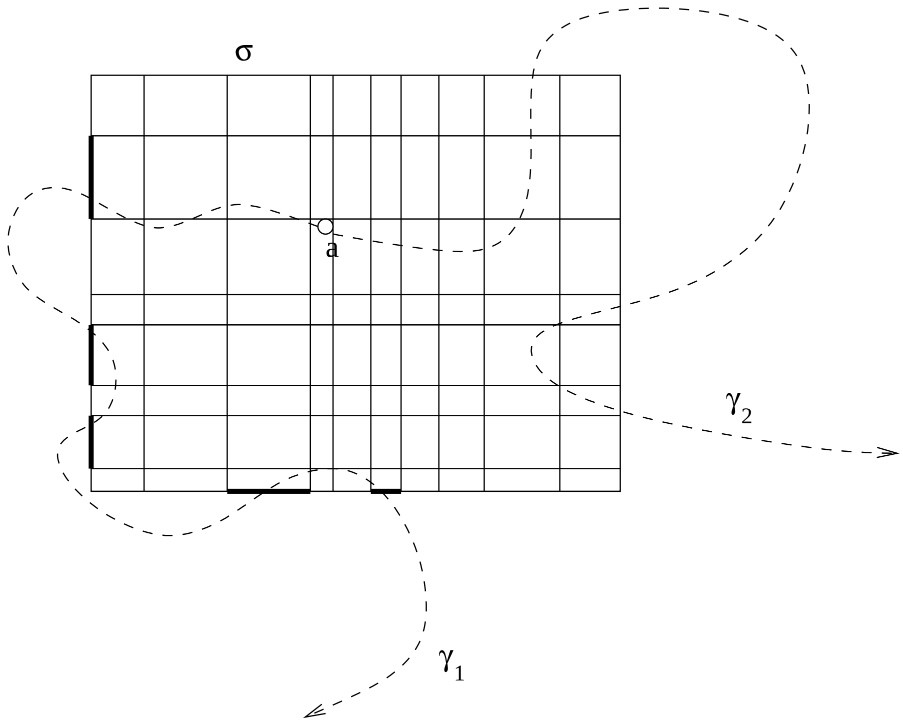


Figure 19: $\gamma_1$, $\gamma_2$ and $\lambda$ for a Jordan Curve

Proceed by contradiction. Assume that every pair $p_i, q_i$ can be connected in $\mathfrak{J}^c$ by a polygonal chain. As we have done before, we can construct a grating $\mathfrak{G}^+$ out of $\mathfrak{G}$ on which every one of these polygonal chains is a 1-chain (we do this simply by adding more and more lines until every edge of the polygonal chains lie on a line. We call the polygonal chain between $p_i$ and $q_i$ $\mu_i$. Therefore, we have $n$ 1-chains, $\mu_1, \mu_2, \ldots, \mu_n$, one for each pair of points. Define $\mu = \sum_{i=1}^{n} \mu_i$.It follows that, $\partial(\lambda) = \partial(\mu) = \{p_1, q_1, p_2, q_2, \ldots, p_n, q_n\}$, which implies that $\partial(\lambda + \mu) = \emptyset$, and $\lambda + \mu$ is a 1-cycle. Note that $\lambda$ does not intersect $\gamma_2$. By our assumption, since each $\mu_i$ does not intersect $\mathfrak{J}$, $\mu$ does not intersect $\mathfrak{J}$, and therefore does not intersect $\gamma_2$.

Define $\gamma_1^c = G_1$ and $\gamma_2^c = G_2$. Since $\gamma_1$ and $\gamma_2$ are paths, they are closed. Therefore, $G_1$ and $G_2$ are open. We see that, since $\lambda$ and $\mu$ do not intersect $\gamma_2$, they are in $G_2$. As we have already noted, $\lambda + \mu$ is a 1-cycle, therefore, it is a 1-cycle on $G_2$. By Corollary I, we see that $\lambda + \mu$ is a boundary on $G_2$.

Now consider $\lambda + \mu + \sigma$. We see that $\partial(\lambda + \mu + \sigma) = \partial(\lambda + \mu) + \partial(\sigma) = \emptyset + \emptyset$. Therefore, $\lambda + \mu + \sigma$

is also a 1-cycle. Notice that $\lambda$ consists of all the edges of $\sigma$ that intersect $\gamma_1$. Therefore, $\lambda + \sigma$ is disjoint from $\gamma_1$. Additionally, $\mu$ is, by assumption, disjoint from $\mathfrak{J}$ and therefore from $\gamma_1$. It follows that $\lambda + \mu + \sigma$ is disjoint from $\gamma_1$, and therefore a 1-cycle on $G_1$. Again using Corollary I, $\lambda + \mu + \sigma$ must be a boundary on $G_1$.

It follows that $(\lambda + \mu + \sigma) + (\lambda + \mu) = \sigma$ is a boundary on $G_1 \cup G_2$. However,

$$G_1 \cup G_2 = \gamma_1^c \cup \gamma_2^c = (\gamma_1 \cap \gamma_2)^c = \{a, b\}^c.$$

Since we have proved the Jordan Curve Theorem for rectangles, we know that $\sigma$ is the boundary for the subset of the plane on the "inside" and the subset of the plane on the "outside" (both of these used in the conventional sense). However, we choose $\sigma$ such that $a$ was on the inside of $\sigma$, and $b$ was on the outside. Therefore, $\sigma$ cannot be the boundary for the subset of the plane that consists of the plane minus $a$ and $b$. This is our desired contradiction, so at least one $p_i, q_i$ cannot be connected without intersecting $\mathfrak{J}$. Therefore, by the Polygonal Chain Theorem, $\mathfrak{J}$ divides the plane into at least two unconnected parts.

- **$\mathfrak{J}$ divides the plane into at most two sections.**

  We know from the previous argument that $\mathfrak{J}$ divides the plane into at least two parts. Therefore, we know that we can select $x, y, z \in \mathfrak{J}^c$ such that $x$ and $y$ are divided by $\mathfrak{J}$, and $y$ and $z$ are divided by $\mathfrak{J}$. That is any polygonal path between $x$ and $y$ intersects $\mathfrak{J}$, and the same is true for $y$ and $z$. We now wish to show that $x$ and $z$ are not divided by $\mathfrak{J}$, that is, they are connected in $\mathfrak{J}^c$.

  Let $a, b \in \mathfrak{J}$. Just as in the previous section of the proof, $a$ and $b$ divide $\mathfrak{J}$ into two paths, $\gamma_1$ and $\gamma_2$. By Theorem 25, neither $\gamma_1$ or $\gamma_2$ divides the plane. Therefore, we can construct a grating $\mathfrak{H}$ such that $x, y, z$ are 0-simplexes in $\mathfrak{H}$ and there exists 1-chains $\lambda_1, \lambda_2, \mu_1$ and $\mu_2$ such that

$$\begin{aligned} \partial(\lambda_1) = \partial(\lambda_2) &= \{x, y\} \\ \partial(\mu_1) = \partial(\mu_2) &= \{y, z\}, \end{aligned}$$

  $\mu_1$ and $\lambda_1$ do not intersect $\gamma_1$, and $\mu_2$ and $\lambda_2$ do not intersect $\gamma_2$.

  Now, let $G_1 = \gamma_1^c$ and $G_2 = \gamma_2^c$. Assume $\lambda_1 + \lambda_2$ was a boundary in $G_1 \cup G_2$. Then by Alexander's Lemma, since $\lambda_1 \sim \lambda_2$ in $G_1 \cup G_2$ and $\partial(\lambda_1) = \partial(\lambda_2) = \{x, y\}$, $x \sim y$ in $G_1 \cap G_2$. By Theorem 22, this means that $x$ and $y$ can be connected by a polygonal chain in $G_1 \cap G_1 = \mathfrak{J}^c$. This contradicts our assumption that $\mathfrak{J}$ divided $x$ and $y$. Therefore, $\lambda_1 + \lambda_2$ is not a boundary in $G_1 \cup G_2$. An identical argument tells us that $\mu_1 + \mu_2$ cannot be a boundary in $G_1 \cup G_2$. Notice at this point that although $\lambda_1 + \lambda_2$ and $\mu_1 + \mu_2$ are not boundaries, they are 1-cycles in $G_1 \cup G_2$.

  By Corollary II, since $\lambda_1 + \lambda_2$ and $\mu_1 + \mu_2$ are not boundaries in the complement of $F = \gamma_1 \cup \gamma_2$, $\lambda_1 + \lambda_2 \sim \mu_1 + \mu_2$ in $G_1 \cup G_2$. Therefore, $(\lambda_1 + \lambda_2) + (\mu_1 + \mu_2)$ is a boundary in $G_1 \cup G_2$. Note that

$$\partial(\lambda_1 + \mu_1) = \partial(\lambda_2 + \mu_2) = x + z,$$

and that $\lambda_1 + \mu_1 \subset G_1$ and $\lambda_2 + \mu_2 \subset G_2$. Therefore, by Alexander's Lemma, $x \sim z$ in $G_1 \cap G_2$. Therefore, $x$ and $z$ can be connected in $\mathfrak{J}^c$, so $\mathfrak{J}$ can divide the plane into at most two parts. Combining the two sections, we therefore see that $\mathfrak{J}$ divides the plane into two parts.

- **$\mathfrak{J}$ forms the boundary of both parts.**

  Let us denote the two disjoint sets that $\mathfrak{J}$ divides the plane into by $R_1$ and $R_2$. Since $\mathfrak{J}$ is a Jordan curve, it is compact by Theorem 8 and closed by by Theorem 7. Therefore, $R_1 \cup R_2$ must be open. Since $R_1$ and $R_2$ are disjoint, this implies that $R_1$ and $R_2$ are both open. Therefore, they do not contain their boundaries by Theorem 5. Therefore, since $R_1$, $R_2$ and $\mathfrak{J}$ exhaust the plane, the boundary of $R_1$, denoted $b(R_1)$ must be a subset of $\mathfrak{J} \cup R_2$, and $b(R_2) \subseteq \mathfrak{J} \cup R_1$. However, if an element of $b(R_1)$ was in $R_2$, then a point of $R_2$ would be near $R_1$. This implies that there exists an element of $R_2$ for which every neighborhood contained at point of $R_1$, and therefore was not contained in $R_2$. This contradicts $R_2$ being open, so $b(R_1)$ must be disjoint from $R_2$, and the argument tells us that $b(R_2)$ is disjoint from $R_1$. Therefore, $b(R_1) \subseteq \mathfrak{J}$ and $b(R_2) \subseteq \mathfrak{J}$.

  Let $a \in \mathfrak{J}$. In proving that $\mathfrak{J}$ divided the plane into at least two parts, we showed that about any $a \in \mathfrak{J}$, there existed an arbitrarily small square $\sigma$ that contained points in both parts of $\mathfrak{J}^c$. Therefore, for any square $\sigma$, there exists $r_1 \in R_1$ and $r_2 \in R_2$ such that $r_1, r_2 \in \sigma$. Therefore, $R_1$ and $R_2$ are near $a$ for all $a \in \mathfrak{J}$. Therefore, all of $\mathfrak{J}$ forms the boundary of $R_1$ and $R_2$.

- **One of $R_1$ and $R_2$ is bounded, the other is unbounded.**

  By Theorem 8, $\mathfrak{J}$ is compact. By the Bolzano-Weierstrass Theorem, this means that $\mathfrak{J}$ is bounded. Therefore, by Definition 10, there exists an open ball containing $\mathfrak{J}$. Since $R_1$, $R_2$, and $\mathfrak{J}$ exhaust the plane, exactly one of $R_1$ and $R_2$ contains points outside this open ball. This subset of the plane is unbounded. The other subset is contained inside the open ball, and is therefore bounded.

This concludes the proof of the Jordan Curve Theorem. □

We have therefore shown that, for any closed curve, the curve divides the plane into two distinct, disconnected regions that possess different properties: one is bounded and the other is not. This is a very simple statement, yet as has been demonstrated its proof is anything but. Alternative methods for proving this theorem exist, many based on analytical rather than combinatorial methods. These other techniques have the advantage that they can be expanded to include high dimensional versions of the Jordan Curve Theorem, something that this proof technique is incapable of.

Now let us return to a topic touched upon in the introduction: space filling curves. It turns out that these curves do not divide the plane. This is because space filling curves are not Jordan curves. The quickest way to see this is to consider the nearness of points on the curve. For example, if you took a Jordan curve and tried to topologically transform it into a Peano curve, you would fail. A Peano curve occupies every point in some region of the plane. Therefore, in the deformation, points that are not near each other in the original curve would have to be moved near each other, in the sense of Definition 2. Once we know that space filling curves are not Jordan curves, we can see way the proof technique used to prove that Jordan curves divide the plane fails for the space filling version. The problem lies in the lemma used to prove Alexander's Lemma. With a space filling curve, you cannot find some $\epsilon > 0$ which separates different halves of the plane. Without this, Alexander's Lemma cannot be applied, and so the Jordan Curve Theorem itself fails.

It is hoped, after reading this paper, that a more complete grasp of the concepts in topology and combinatorial graph theory has been obtained. The Jordan Curve Theorem itself is a very interesting piece of mathematics, with a great deal of application. In addition, its proof provides excellent opportunity to bring to bear several important fields of mathematics.

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