The Wise Yenta Theorems: Managing Marriage and Divorce

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1 Introduction

When my father, Professor Ben Schumacher, was collaborating on a paper about quantum entanglement (see [3]), he modeled part of the problem using a metaphor he called the Wise Yenta problem. In the example, a yenta (or gossipy old woman) was given the task of acting as a matchmaker for a village. Wise as she was, she want to marry off all the girls in the village to suitable men. When is this is possible? And what happens if none of her matches work out, and she has to help a bunch of recent divorcées find love?

It turns out that both of these problems can be solved using graph theory, a versatile branch of mathematics that (among other things) can be used to model complex social relationships. This paper will give an introduction to this field through following the Wise Yenta narrative, which we will introduce in the next section.

2 The Yenta and the Marriage Problem

Once upon a time, there was an old woman who everyone called the Wise Yenta. The things she loved best in the world were gardening, mathematics, and helping people in need. One day, she went to visit a nearby village to buy some new seeds. This particular village had no matchmaker, and there were n eligible young girls ready to be married off. Desperate to find them husbands, the families of these girls had begun to fight amongst themselves, and the whole village was in turmoil. Seeing the distress it was causing, the Wise Yenta stepped in and offered to serve as matchmaker herself.

The problem proved more complicated than she expected. To begin with, there were only a limited number of young men available to marry these girls. Furthermore, because the Wise Yenta was a stranger to the area, the families each provided a list of vetted candidates and refused to consider any other offers. With these restrictions, it was unclear whether some of the girls could marry at all, but the Yenta was undaunted. Surely mathematics could solve this thorny problem!

The Wise Yenta's dilemma turns out to be a variant of a classic thought experiment known as the Marriage Problem, as shown in Problem 2.0.1:

Problem 2.0.1. Suppose there exists a finite set of m girls G, who between them collectively know the members of a set of boys B (of no particular size). Our goal is to marry them off in such a way that each girl is paired with a boy she knows. Under what conditions does a solution exist?

And luckily for the Yenta, the Marriage Problem was solved in 1935, by mathematician Philip Hall (see [6]). His result became known as Hall's Marriage Theorem:

Theorem 2.1 (Hall's Marriage Theorem)**.** A solution to the marriage problem exists iff each subset of k girls in G collectively knows at least k boys in B, for $1 \leq k \leq m$.

In the next section, we will work through two classic proofs of this theorem, each using a different mathematical technique. But before we can begin proving anything, we will need some foundational knowledge of graph theory. The following definitions, found in [6] and [1], will serve to get us off the ground:

Definition 2.2. A *graph* is an ordered pair of disjoint sets $G = G(V, E)$, where $E \subseteq V \times V$. Elements of V are known as *vertices* and elements of E are known as *edges*.

There are actually two types of graphs, defined by the properties of their set of edges. An *undirected graph* has the symmetric and antisymmetric properties: that is, for every $a, b \in V$, if the edge $(a, b) \in E$, then the edge $(b, a) \in E$ and $(a, b) = (b, a)$. In a *directed graph*, this is not the case. However, for the purposes of this paper, all graphs examined will be undirected graphs, and we will represent the edge (a, b) by simply writing ab.

Graphs can also be represented visually, where each vertex $(a, b, etc.)$ is represented by a point in space and each edge ab is a line segment with vertices a and b as endpoints (as in Figure 1).

Definition 2.3. Two vertices of a graph $a, b \in V$ are *adjacent* if there exists an edge $e \in E$ such that $e = ab$.

Notice, now, how simple it is to model the marriage problem using graph theory. The people involved can be represented by vertices, and two vertices are adjacent if they know each other. However, also notice that the vertices will fall into two distinct sets (boys and girls) and that the only edges possible will be between these two sets. This is an example of a specific type of graph known as a bipartite graph:

Definition 2.4. A *bipartite graph* is a graph $G(V, E)$ where V is the disjoint union of two sets V_1 and V_2, and for every edge $ab \in E$, $a \in V_1$ and $b \in V_2$ (as in Figure 2). Bipartite graphs are typically written $G = G(V_1, V_2, E)$.

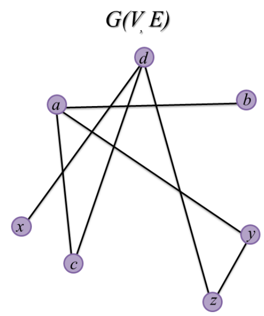

Figure 1: The visual representation of a graph.

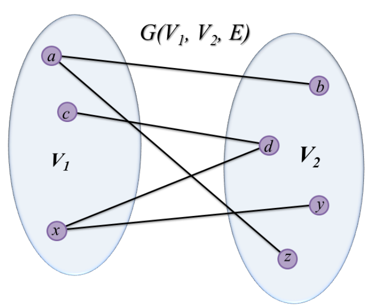

Figure 2: The visual representation of a bipartite graph, with the sets V_1 and V_2 indicated.

It is also possible to consider the pieces of a graph as opposed to the whole. We can even combine these together into new graphs, known as subgraphs.

Definition 2.5. A *subgraph* of a graph $G = G(V, E)$ is a graph $G{*}(V{*}, E{*})$, where $V{*} \subseteq V$ and $E{*} \subseteq E$.

Examining subgraphs can be useful for a lot of different problems. For the Yenta's, a specific type of subgraph is especially useful:

Definition 2.6. A *complete matching* in a bipartite graph $G = G(V_1, V_2, E)$ is a subgraph of G in which every vertex in V_1 is adjacent to at least one vertex in V_2. (Note, however, that there may be vertices in V_2 that are not adjacent to vertices in V_1).

Notice that any solution to the Marriage Problem will be a complete matching: every girl in V_1 will be paired with a boy in V_2. For the first proof of Theorem 2.1 we will examine, nothing further is needed. However, the second proof requires one more definition from set theory, that of a transversal:

Definition 2.7. Let K be a non-empty finite set, and let $\mathbf{F}$=$S_1, \ldots, S_m$ be a collection of non-empty (not necessarily distinct) subsets of K. A *transversal* of $\mathbf{F}$ is a set of m distinct elements of K, one chosen from each of the sets S_i.

3 Two Proofs of the Marriage Theorem

We begin with the first of two classic proofs of Theorem 2.1, the proof attributed to Halmos and Vaughn (see [6]). It uses complete induction, a typical way to prove graph theory concepts, and can be stated easily in 'marriage' terminology. The advantage of this is that the proof is very easy to understand, even by people without much background in graph theory: in fact, it is almost unnecessary to know what a graph is! However, the same proof can be written in terms of adjacent vertices and finding a complete matching.

Halmos-Vaughn Proof of the Marriage Theorem.

($\Longrightarrow$) Suppose that a solution to the marriage problem exists; that is, that the Wise Yenta can arrange marriages such that every girl is married to a boy she knows. Then each girl knows at least one boy: namely, the one she is married to. Thus, any two girls will collectively know at least the two boys they were paired with, any three girl will know at least three boys, and so on, all the way up to m. Thus, if a solution exists, each subset of k girls collectively knows at least k boys, for $1 \leq k \leq m$.

($\Longleftarrow$) Let $m \in \mathbb{N}$. Assume that for some set G of m girls, every subset of k girls collectively know at least k boys for $1 \leq k \leq m$. We shall proceed by induction on m. Note that the base case is trivial: if $|G| = 1$, that girl must know at least one boy in order to marry.

Now suppose that for all of girls G_t of size t, $1 \leq t < m$, Hall's marriage theorem holds true–in other words, if every subset of size k collectively knows at least k boys, then the Wise Yenta can marry them off. Now consider again our set of size m. Given our original assumption about G, we can now split this section of the proof into two subcases:

Case (i) Suppose that for all $k < m$, each subset of k girls collectively knows at least $k+1$ boys. The Wise Yenta can now marry one of the girls, g_m, to a boy she knows, b_m, leaving the other $m-1$ girls unmatched. Note now that every subset of k unmatched girls collectively knows at least k eligible boys: if they knew $k+1$ including b_m, they now know k. Furthermore, $m-1 < m$; thus, by our induction hypothesis, the Wise Yenta can marry off the rest of the remaining girls to boys they know.

Case (ii) Suppose that for some $k < m$, there exists a subset of k girls G_k* that collectively knows exactly k boys. By our induction hypothesis, the Wise Yenta can marry off these girls to those boys, leaving $m - k$ girls unmatched. Notice that among those remaining girls, every subset of h girls G_h* must know at least h eligible boys–otherwise, the girls in the subset $G_k * \cup G_h*$ would know fewer than $h + k$ boys, violating our original assumption. Thus, since $m - k < m$, the remaining girls can all be married off to boys they know.

In either case, a solution to the marriage problem exists. Thus, if Hall's Marriage Theorem is true for $t < m$, it is also true for m.

Therefore, by mathematical induction, if every subset of k girls in G collectively know at least k boys, for $1 \leq k \leq m$, then a solution to the marriage problem exists. Therefore, we have proven Hall's Marriage Theorem.

∎

The second proof of the Marriage Theorem is attributed to Rado (see [6]) and uses set theory. It has a few advantages compared to that of Halmos-Vaughn: not only does it avoid splitting the proof into cases, it also suggests an algorithm for finding a solution to the Marriage Problem (rather than just proving a solution must exist). However, it is almost impossible to state in marriage terminology, requiring us to restate Theorem 2.1 in a different way:

Theorem 3.1. Let G be a non-empty finite set, and let $\mathbf{F}$=$S_1, \ldots, S_m$ be a collection of non-empty subsets $S_i \subseteq G$. Then $\mathbf{F}$ has a transversal if and only if the union of any k of the subsets S_i contains at least k elements $(1 \leq k \leq m)$.

Though it looks very different, this is the same as Hall's Marriage theorem. Given a set of girls G, we can think of each girl as the name of a subset S_i (with each boy she knows being an element of S_i). Finding a transversal of G, then, is logically equivalent to selecting one boy from each girl's list for her to marry–in other words, finding a solution. Since the condition for a transversal to exist is the same as before, all that remains is to prove the theorem.

Rado Proof of the Marriage Theorem.

($\Longrightarrow$) Suppose that $\mathbf{F}$ has a transversal. Then each set S_j has at least one element distinct from all the others. Thus, the union of any k of the those S_js will contain at least k elements, for $1 \leq k \leq m$.

($\Longleftarrow$) Suppose that the union of any k of the subsets S_i contains at least k elements $(1 \leq k \leq m)$. Note that if the S_is were all disjoint singleton sets, the proof would be trivial: the transversal would just be the union of all the S_i. Also note that if any of the sets (without loss of generality, call it S_1) had more than one element, and if it were always possible to remove one of the elements of S_1 without violating our original assumption, we could repeat the process and reduce the problem to a case of distinct singleton sets (through relabelling if necessary).

To prove that we can in fact remove one of the elements of S_1, we shall proceed by contradiction. Suppose there exists an S_1 incapable of being reduced to a singleton set: that is, for some distinct $x, y \in S_1$, the removal of either x or y from S_1 violates our original assumption. Then there exist two sets $A, B \subseteq \{2, 3, \ldots, m\}$ such that, for $P = \bigcup_{j \in A} S_j \cup (S_1 \setminus \{x\})$ and $Q = \bigcup_{k \in B} S_k \cup (S_1 \setminus \{y\})$, $|P| \leq |A|$ and $|Q| \leq |B|$.

Notice, then, that because unions can be taken in any order and $x \neq y$:

$$\begin{aligned} P \cup Q &= \left(\bigcup_{j\in A} S_j \cup (S_1 \setminus \{x\})\right) \bigcup \left((S_1 \setminus \{y\}) \cup \bigcup_{k\in B} S_k\right) \\ &= (S_1 \setminus \{x\}) \cup (S_1 \setminus \{y\}) \bigcup \left(\bigcup_{j\in A\cup B} S_j\right) \\ &= S_1 \cup \left(\bigcup_{j\in A\cup B} S_j\right). \end{aligned}$$

Thus, we can conclude that $|P \cup Q| = \left|S_1 \cup \bigcup_{j\in A\cup B} S_j\right|$.

Notice, also, that because unions and intersections distribute over each other and $S \subseteq S \cup T$ for any sets S, T:

$$\begin{aligned} P \cap Q &= \left(\bigcup_{j\in A} S_j \cup (S_1 \setminus \{x\})\right) \cap \left(\bigcup_{k\in B} S_k \cup (S_1 \setminus \{y\})\right) \\ &= \left(\bigcup_{j\in A} S_j \cup (S_1 \setminus \{x\}) \cap (\bigcup_{k\in B} S_k)\right) \cup \left(\bigcup_{j\in A} S_j \cup (S_1 \setminus \{x\}) \bigcap (S_1 \setminus \{y\})\right) \\ &\supseteq \left(\bigcup_{j\in A} S_j \cup (S_1 \setminus \{x\})\right) \cap \left(\bigcup_{k\in B} S_k\right) \\ &= \left(\bigcup_{j\in A\cap B} S_j\right) \cup \left((S_1 \setminus \{x\}) \cap (\bigcup_{k\in B} S_k)\right) \\ &\supseteq \bigcup_{j\in A\cap B} S_j. \end{aligned}$$

Hence, $|P \cap Q| \geq \left|\bigcup_{j\in A\cap B} S_j\right|$.

This gives us all the pieces we need to reach a contradiction, proceeding as follows:

$$
\begin{aligned}
|A| + |B| &\geq |P| + |Q| && (1)\\
&= |P \cup Q| + |P \cap Q| && (2)\\
&\geq \left| S_1 \cup \bigcup_{j \in A \cup B} S_j \right| + \left| \bigcup_{j \in A \cap B} S_j \right| && (3)\\
&\geq (|A \cup B| + 1) + |A \cap B| && (4)\\
&= |A| + |B| + 1 && (5)
\end{aligned}
$$

Inequality (1) follows from our definition of P and Q, Equality (2) from a well-known fact in set theory known as the Principle of Inclusion and Exclusion (see Chapter 3 of [4]). Inequality (4) follows from Hall's condition (i.e., that every union of k subsets contains at least k elements), and Inequality (5) follows from the same principle as (2). This leads to a contradiction, since $|A| + |B| < |A| + |B| + 1$ for finite sets. Therefore, every S_i can be reduced to a singleton set without violating Hall's condition, and hence **F** has a transversal.

Therefore, **F** has a transversal if and only if the union of any k of the subsets S_i contains at least k elements $(1 \leq k \leq m)$. ∎

Notice that both proofs insist that the set of girls be finite, and this is indeed necessary: the proofs fail when considering infinite sets. Interestingly, though, the proofs fail in different ways. The Halmos-Vaughn proof fails because induction on the size of infinite sets simply does not make any sense. The Rado proof, on the other hand, fails because of cardinality: the crucial contradiction that makes the reduction process valid is no longer contradictory when dealing with infinite sets, and thus a transversal cannot be found. In both cases, however, we end up at the same conclusion: Hall's Marriage Theorem only applies to finite sets, and tells us nothing about the infinite case.

4 The Yenta and the Dating Divorcées

Now let us return to the story of the Wise Yenta. Having determined that she can indeed pair off all the young ladies to men they have met, she goes through Rado's reduction process and manages to arrange matches for all her clients. The families are delighted and host a massive n-tuple wedding, and the Yenta goes back to her gardening, satisfied that she has brought a little happiness into the world.

Unfortunately, the fact that two people are acquainted with one another does not imply that they will be compatible as a couple. A few months later, the Yenta returns to the village and discovers to her dismay that her arranged marriages, far from being blissful, have all ended in messy divorces. Horrified that she has caused so much unhappiness, the Wise Yenta decides to make amends for her poor choices. She approaches the families again, this time offering to help the recent divorcées find love for themselves. She plans to host a series of get-togethers to which subsets of the young people will be invited, allowing them to mingle and meet each other–but, wishing to avoid any awkwardness, she will carefully organize the guest lists to keep estranged couples apart. The families reluctantly agree, and the Wise Yenta turns once again to mathematics to help her figure out the logistics.

To begin with, like any good mathematician, she tries to get a feel for the problem by considering a smaller example. She begins with a subset of five couples: Alfred and Amelia, Edgar and Evelyn, Ivan and Irene, Otto and Olivia, and Ulrich and Ursula (see Figure 3).

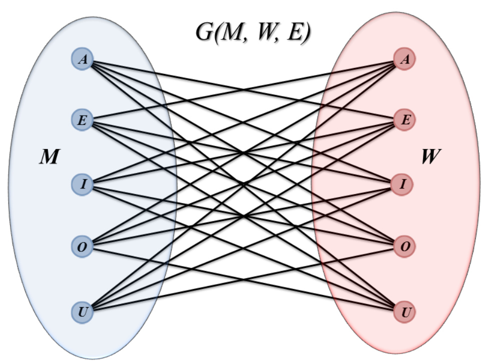

Figure 3: The graph of the admissible matches in the Five Couples example. M is the set of men and W is the set of women.

At first, the Yenta considers sending all the eligible couples on a series of solo dates, but she quickly abandons that idea due to fiscal concerns. Since she is paying for all the parties herself, she decides to find an arrangement with as few parties as possible. Also, she decides to avoid redundancy and make sure that each eligible couple meets exactly one time. After playing around for a while, she finds three arrangements to her liking:

- **The Harem:** In her first arrangement, the Yenta throws five parties. To each, she invites one man, and all the women except the man's ex-wife (see Figure 4).

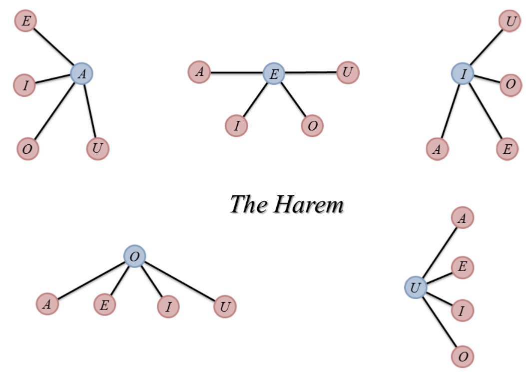

Figure 4: The Harem Arrangement for the Five Couples example.

- **Belle of the Ball:** The second arrangement is a reflection of the first: five parties, each attended by one woman and all men except her ex-husband (see Figure 5).

- **The Double-Blind Date:** In her third arrangement, the Yenta organizes five parties with two men and two women attendees each (see one example in Figure 6).

All three of these arrangements have a number of things in common. First, the smallest numbers of parties she could find was five–the same as the number of divorced couples. Second, each party is attended by the same number of men and the same number of women–and the product of these numbers is $1 * 4 = 4 * 1 = 2 * 2 = 4$. Furthermore, the number of parties each man attends is the *same number* as the number of men at each party, and a similar fact is true for the women. Finally, these party groupings are formed in a specific manner: if one were to swap the sets of men or women from any two parties, those two subsets would each contain one–and only one–divorced couple. As it turns out, these properties also characterize the parties that can be arranged for n couples; this leads to the following theorem, whose statement and proof can be found in [5]:

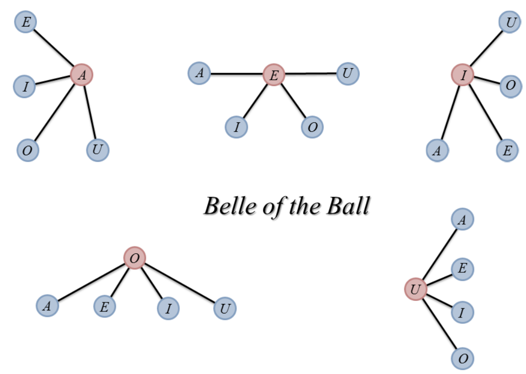

Figure 5: The Belle of the Ball Arrangement for the Five Couples example.

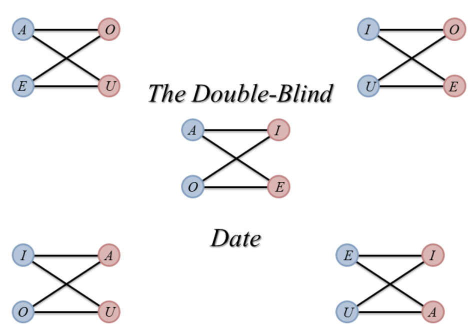

Figure 6: The Double-Blind Date Arrangement for the Five Couples example.

Theorem 4.1 (The Divorce Theorem)**.** Suppose the Wise Yenta wishes to arrange a series of parties for n divorced couples, $n \geq 2$, where each eligible couple meets exactly once and estranged couples never meet. The fewest number of parties she can host under these conditions is n.

Furthermore, given a set of n such parties, there exist $r, s \in \mathbb{Z}^+$ such that

a) $rs = n - 1$,

b) There are r men and s women attending each party,

c) Each man attends exactly r parties and each woman attends exactly s parties, and

d) Given any two distinct parties, the union of the men at the first party and the women at the second party will contain exactly one estranged couple.

Before we can begin proving this theorem, we will need a few additional definitions, also found in [5]:

Definition 4.2. Let $B = G(V_1, V_2, E)$ be a bipartite graph. A *biclique* of B is an ordered pair (X, Y), where $X \subseteq V_1$, $Y \subseteq V_2$, and $X \times Y \subseteq E$.

Definition 4.3. Let $B = G(V_1, V_2, E)$ be a bipartite graph. Two bicliques of B (X_1, Y_1) and (X_2, Y_2) are said to be *disjoint* if $(X_1 \times Y_1) \cap (X_2 \times Y_2) = \emptyset$.

Definition 4.4. Let $B = G(V_1, V_2, E)$ be a bipartite graph. A *biclique partition* of B is a collection of bicliques of B $(X_1, Y_1), (X_2, Y_2), \ldots, (X_t, Y_t)$ such that (X_i, Y_i) and (X_j, Y_j) are disjoint for all $i \neq j$, and $E = (X_1 \times Y_1) \cup (X_2 \times Y_2) \cup \ldots \cup (X_t \times Y_t)$.

Notice that for each $uw \in E$, $(\{u\}, \{w\})$ is a biclique of B. Thus, the collection of all such bicliques is a biclique partition by definition, so B always has at least one. Often, however, B has more than one biclique partition, and $\bigcup_{uw \in E} (\{u\}, \{w\})$ is not the most efficient way to partition the graph. The next two definitions give us a way of identifying and quantifying a partition's efficiency.

Definition 4.5. Let $B = G(V_1, V_2, E)$ be a bipartite graph. The *biclique partition number* of B is the smallest number t such that there exists a biclique partition $(X_1, Y_1), (X_2, Y_2), \ldots, (X_t, Y_t)$ of B.

Definition 4.6. Let $B = G(V_1, V_2, E)$ be a bipartite graph. An *exact biclique partition* of B is a biclique partition $(X_1, Y_1), (X_2, Y_2), \ldots, (X_t, Y_t)$ of B where t is the biclique partition number of B.

From here, we can translate the Wise Yenta's problem into graph-theoretical terms. In this situation, $B = G(M, W, E)$, where M is the set of men, W is the set of women, and E is the set of all eligible couples. Each individual party the Yenta hosts is a biclique, the set of all parties she hosts is a biclique partition, and the fewest number of parties she can host is the biclique partition number. The Yenta's eventual goal is to create an exact biclique partition of B.

Again following [5], we will use elementary linear algebra techniques to prove Theorem 4.1. We can do this by defining a matrix that represents our graph, called an adjacency matrix.

Definition 4.7. Let $G = (V, E)$ be a graph, where $V = \{v_1, v_2, ..., v_n\}$. The *adjacency matrix* of G is the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

This definition is general and will work for any graph. However, notice that in the case of a bipartite graph, the adjacency matrix will always be divided into four blocks: two of them (the ones with only elements of V_1 or only V_2) will contain only zeroes, and the remaining two will be transposed versions of each other. Thus, it is often more useful to use an alternate matrix representation, the reduced adjacency matrix:

Definition 4.8. Let $G = (V_1, V_2, E)$ be a bipartite graph, where $V_1 = \{u_1, u_2, ..., u_m\}$ and $V_2 = \{w_1, w_2, ..., w_k\}$. The *reduced adjacency matrix* of G is the $m \times k$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } u_i w_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

From this point on, we have all the definitions necessary to prove Theorem 4.1, which we shall do in Section 5.

5 Proving the Divorce Theorem

Before we can prove Theorem 4.1, we will first need to prove a series of technical lemmas. Two of them, Lemma 5.4 and Lemma 5.6, are needed to prove the theorem itself, and the rest of them help to prove these lemmas. The notation defined at each stage will be used throughout the rest of the paper unless explicitly stated otherwise.

Let $M = \{m_1, \ldots, m_k\}$ and $W = \{w_1, \ldots, w_l\}$. Let $B = G(M, W, E)$, where $E \subseteq M \times W$. The reduced adjacency matrix of B is the $k \times l$ matrix $A = [a_{ij}]$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } m_i w_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

For any $R \subseteq M$, we shall define the column vector

$$\vec{R} = \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix}$$

where

$$r_i = \begin{cases} 1 & \text{if } m_i \in R \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for any $S \subseteq W$, we shall define the column vector

$$\vec{S} = \begin{bmatrix} s_1 \\ \vdots \\ s_l \end{bmatrix}$$

where

$$s_i = \begin{cases} 1 & \text{if } w_i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\vec{S}^T = [s_1 \ \ldots \ s_l]$. Notice that $\vec{R}\vec{S}^T = [b_{ij}]$ is a $k \times l$ matrix, and that

$$b_{ij} = \begin{cases} 1 & \text{if } r_i = s_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

(This is pretty easy to see, since $\vec{R}$ is a single column and $\vec{S}^T$ is a single row.)

Notice, also, that if G is a bipartite graph $G(M, W, R \times S)$, then $\vec{R}\vec{S}^T$ is the reduced adjacency matrix of G. From here, we proceed to the first of our lemmas:

Lemma 5.1. Let b_{ij} be the ij^{th} entry of $\vec{R}\vec{S}^T$. Then $b_{ij} \leq a_{ij}$ if and only if (R, S) is a biclique of B.

Proof.

($\Longleftarrow$) Suppose that (R, S) is a biclique of B. Then $R \times S \subseteq E$ by definition. Therefore, if $a_{ij} = 0$ for any i, j, $b_{ij} = 0$. If $a_{ij} = 1$, then either $b_{ij} = 0$ or $b_{ij} = 1$ depending on whether $m_i w_j \in R \times S$. Thus, $b_{ij} \leq a_{ij}$ for all i, j.

($\Longrightarrow$) Suppose that for some matrix $G = \vec{R}\vec{S}^T$, $b_{ij} \leq a_{ij}$ for all i, j. Thus, $b_{ij} = 1$ implies that $a_{ij} = 1$. By the definition of a reduced adjacency matrix, the ij^{th} entry is 1 if and only if $m_i w_j$ is in the edge set for the graph. Therefore, all the elements of $R \times S$ are also elements of E. Therefore, by definition, $R \times S \subseteq E$. Since $R \in M$ and $S \in W$ by design, we can now conclude that (R, S) is a biclique of B.

Therefore, $b_{ij} \leq a_{ij}$ for all i, j if and only if (R, S) is a biclique of B. ∎

Now we will use this fact to prove the next lemma, which characterizes a biclique partition by its adjacency matrix, and vice versa.

Lemma 5.2. Let $X_1, X_2, \ldots, X_p \subseteq M$ and $Y_1, Y_2, \ldots, Y_p \subseteq W$. Then $(X_1, Y_1), (X_2, Y_2), \ldots, (X_p, Y_p)$ is a biclique partition of B iff $A = \sum_{l=1}^{p} \vec{X}_l \vec{Y}_l^T$.

Proof.

($\Longrightarrow$) Suppose that $(X_1, Y_1), (X_2, Y_2), \ldots, (X_p, Y_p)$ forms a biclique partition of B. Suppose also that for some i, j, $a_{ij} = 1$. Then $m_i w_j \in E$ and $m_i w_j \in X_s \times Y_s$ for exactly one value $1 \leq s \leq p$. Thus, the ij^{th} entry of $\vec{X}_s \vec{Y}_s^T = 1$, and the ij^{th} entry of $\vec{X}_r \vec{Y}_r^T = 0$ for $r \neq s$. Thus, the ij^{th} entry of $\sum_{l=1}^{p} \vec{X}_l \vec{Y}_l^T = 1 = a_{ij}$.

Now suppose that for some i, j, $a_{ij} = 0$. Then $m_i w_j \notin E$, and there is no r such that $m_i w_j \in X_r \times Y_r$. Thus, the ij^{th} entry of $\vec{X}_r \vec{Y}_r^T = 0$ for all $1 \leq r \leq p$. Thus, the ij^{th} entry of $\sum_{l=1}^{p} \vec{X}_l \vec{Y}_l^T = 0 = a_{ij}$.

Therefore, $a_{ij} =$ the ij^{th} entry of $\sum_{l=1}^{p} \vec{X}_l \vec{Y}_l^T$ for all i, j. Therefore, by definition, $A = \sum_{l=1}^{p} \vec{X}_l \vec{Y}_l^T$.

($\Longleftarrow$) Suppose that $A = \sum_{l=1}^{p} \vec{X_l}\vec{Y_l}^T$. Since all the summands' entries are either 0s or 1s, the entries of each $\vec{X_r}\vec{Y_r}^T$ must be less than or equal to each to the corresponding entries of A. Thus, by Lemma 5.1, each (X_r, Y_r) is a biclique of B.

Now consider an arbitrary edge $m_i w_j \in E$. Since $a_{ij} = 1$, we know that the ij^{th} entry of $\sum_{l=1}^{p} \vec{X_l}\vec{Y_l}^T$ is also 1. And since each $\vec{X_r}\vec{Y_r}^T$ is a $(0,1)$-matrix, there must be a unique s such that the ij^{th} entry of $\vec{X_s}\vec{Y_s}^T = 1$. Thus, $m_i w_j \in X_s \times Y_s$ for exactly one s, and thus is an element of exactly one of the bicliques. Since $m_i w_j$ was arbitrary, this will be true of any edge in E. Thus, $(X_1, Y_1), (X_2, Y_2), \ldots, (X_p, Y_p)$ forms a biclique partition of B.

Therefore, $(X_1, Y_1), \ldots, (X_p, Y_p)$ forms a biclique partition of B iff $A = \sum_{l=1}^{p} \vec{X_l}\vec{Y_l}^T$. ∎

This particular characterization of a biclique partition is useful, but it can be taken even farther; we can use block multiplication of matrices to write A in terms of a matrix product, as the next lemma will show.

Lemma 5.3. $(X_1, Y_1), \ldots, (X_p, Y_p)$ is a biclique partition of B iff $A = XY$, where X is a $k \times p$ $(0,1)$ matrix whose i^{th} column is $\vec{X_i}$, and Y is a $p \times l$ $(0,1)$ matrix whose i^{th} row is $\vec{Y_i}^T$.

Proof.

($\Longrightarrow$) Let $X_1, X_2, \ldots, X_p \subseteq M$ and $Y_1, Y_2, \ldots, Y_p \subseteq W$, and suppose that $(X_1, Y_1), \ldots, (X_p, Y_p)$ is a biclique partition of B. Define $X = [\vec{X_1} \ \cdots \ \vec{X_p}]$ and

$$Y = \begin{bmatrix} \vec{Y_1}^T \\ \vdots \\ \vec{Y_p}^T \end{bmatrix}$$

Then, by using block multiplication of matrices, we find that $XY = \sum_{l=1}^{p} \vec{X_l}\vec{Y_i}^T = A$, the reduced adjacency matrix of B.

($\Longleftarrow$) Conversely, suppose that there exist $k \times p$ and $p \times l$ $(0,1)$ matrices X and Y such that $A = XY$. Define $\vec{X_i}$ to be the i^{th} column of X, and $\vec{Y_i}^T$ to be the i^{th} row of Y. Then by block multiplication of matrices, $A = XY = \sum_{l=1}^{p} \vec{X}_l \vec{Y}_i^T$, and by Lemma 5.2, $(X_1, Y_1), \ldots, (X_p, Y_p)$ is a biclique partition of B.

Therefore, $(X_1, Y_1), \ldots, (X_p, Y_p)$ is a biclique partition of B iff $A = XY$. ∎

Notice that in this scenario, p is the number of distinct bicliques in our biclique partition. The biclique partition number $bp(B)$, therefore, is the smallest value of p such that $A = XY$. By specializing to our particular example of the n divorced couples, $n \geq 2$, we can now prove the first of our pivotal lemmas:

Lemma 5.4. Let $M = \{m_1, \ldots, m_n\}$ be the set of men and $W = \{w_1, \ldots, w_n\}$ be the set of women. Let $B = G(M, W, E)$, where $E = \{m_i w_j \in M \times W | i \neq j\}$. Then $bp(B) = n$.

Proof. By our definition of the matrix X, we know that $bp(B) \geq \text{rank}(X) \geq \min(\text{rank}(X), \text{rank}(Y))$. We also know from linear algebra that $\min(\text{rank}(X), \text{rank}(Y)) \geq \text{rank}(XY) = \text{rank}(A)$ (see Section 3.5 of [2]). Thus, $bp(B) \geq \text{rank}(A)$.

Let $A = J_n - I_n$, where J_n is the $n \times n$ matrix of all 1s, and I_n is the $n \times n$ identity matrix. For any $n \in \mathbb{N}$, we can perform the following calculation:

$$\begin{aligned} \left(\frac{1}{n-1}(J_n - I_n)\right) A &= \left(\frac{1}{n-1}(J_n - I_n)\right)(J_n - I_n) \\ &= I_n \end{aligned}$$

Thus, A is invertible, and by the Fundamental Theorem of Invertible Matrices (see [2]), we know that $\text{rank}(A) = n$. Thus $bp(B) \geq n$. Notice that $(\{m_i\}, \{w_j : i \neq j\})$ is a biclique of B for each $1 \leq i \leq n$ (i.e., the Harem arrangement from our example). Furthermore, $(\{m_i : i \neq j\}, \{w_j :\})$ is a biclique of B for each $1 \leq j \leq n$ (i.e., the Belle of the Ball arrangement). Thus, there exist biclique partitions with size $|M| = |W| = n$, and we know that $bp(B) \leq n$. Thus, $n \leq bp(B) \leq n$, and by the antisymmetric property of $\leq$, $bp(B) = n$. ∎

This lemma tells us that given an exact biclique partition of of the n divorced couples, the matrices X and Y will be of dimension $n \times n$–in other words, they must be square!

At this point, we move onto the second act of the proof. After a brief interlude in which we prove a technical fact about determinants, we shall then use this to prove the pivotal Lemma 5.6, where most of the work of proving Theorem 4.1 is accomplished.

Lemma 5.5. Suppose that for $A = XY$, X is an $n \times k$ and Y is a $k \times n$ matrix. Then $det(I_n + XY) = det(I_k + YX)$.

Proof.

$$\begin{aligned}
\begin{bmatrix} I_k + YX & 0 \\ X & I_n \end{bmatrix} \begin{bmatrix} I_k & Y \\ 0 & I_n \end{bmatrix} &= \begin{bmatrix} (I_k)^2 + I_k(YX) + 0 & Y + YX(Y) + 0 \\ X + 0 & XY + (I_n)^2 \end{bmatrix} \\
&= \begin{bmatrix} I_k + YX & Y + YXY \\ X & XY + I_n \end{bmatrix} \\
&= \begin{bmatrix} I_k + YX & Y(I_n + XY) \\ X & I_n + XY \end{bmatrix} \\
&= \begin{bmatrix} I_k^2 + YX & 0 + Y(I_n + XY) \\ X & I_n^2 + I_n(XY) \end{bmatrix} \\
&= \begin{bmatrix} I_k & Y \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_k & 0 \\ X & I_n + XY \end{bmatrix}
\end{aligned}$$

Since these two matrix products are equal, their products' determinants must also be equal. Therefore:

$$\begin{aligned}
det\left(\begin{bmatrix} I_k + YX & 0 \\ X & I_n \end{bmatrix} \begin{bmatrix} I_k & Y \\ 0 & I_n \end{bmatrix}\right) &= det\left(\begin{bmatrix} I_k & Y \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_k & 0 \\ X & I_n + XY \end{bmatrix}\right) \\
det\begin{bmatrix} I_k + YX & 0 \\ X & I_n \end{bmatrix} det\begin{bmatrix} I_k & Y \\ 0 & I_n \end{bmatrix} &= det\begin{bmatrix} I_k & Y \\ 0 & I_n \end{bmatrix} det\begin{bmatrix} I_k & 0 \\ X & I_n + XY \end{bmatrix} \\
det(I_k + YX) * det(I_n) * det(I_k) * det(I_n) &= det(I_k) * det(I_n) * det(I_k) * det(I_n + XY) \\
det(I_k + YX)(1)(1)(1) &= (1)(1)(1)det(I_n + XY) \\
det(I_k + YX) &= det(I_n + XY)
\end{aligned}$$

∎

Lemma 5.6. Let X and Y be $n \times n$ $(0,1)$-matrices with $XY = J_n - I_n$ and $n \geq 2$. Then $XY = YX$, and there exist $r, s \in \mathbb{Z}^+$ with $rs = n - 1$ such that $XJ_n = J_nX = rJ_n$ and $YJ_n = J_nY = sJ_n$.

Proof. Let $\vec{X_i}$ = the i^{th} column of X and $\vec{Y_i}^T$ = the i^{th} row of Y. By definition, we know that $tr(XY) = tr(J_n - I_n) = 0 + 0 + \ldots + 0 = 0$. Furthermore, $tr(XY) = tr(YX)$ for any matrix product XY (see Section

3.2 of [2]). The definition of trace tells us that $tr(YX) = \sum_{l=1}^{n} \vec{Y_i}^T \vec{X_i}$. Thus, $\sum_{l=1}^{n} \vec{Y_i}^T \vec{X_i} = 0$, and since these are all $(0,1)$-matrices, $\vec{Y_i}^T \vec{X_i} = 0$ for all $1 \le i \le n$.

Let e be the $n \times 1$ vector of all 1s. Then $J_n = ee^T$. Fix $i, j \le n$. For $i \neq j$, we have

$$\begin{aligned} A &= J_n - I_n \\ &= ee^T - I_n \\ &= XY \\ &= \sum_{l=1}^{n} \vec{X_l} \vec{Y_l}^T. \end{aligned}$$

Thus, $ee^T - I_n = \sum_{l=1}^{n} \vec{X_l} \vec{Y_l}^T$. Now, adding I_n and subtracting $\sum_{l \neq i,j} \vec{X_l} \vec{Y_l}^T$ from both sides of the equation, we can conclude that $ee^T - \sum_{l \neq i,j} \vec{X_l} \vec{Y_l}^T = I_n + \vec{X_i} \vec{Y_i}^T + \vec{X_j} \vec{Y_j}^T$.

Notice rank$(ee^T) = 1$, and that each $\vec{X_k} \vec{Y_k}^T$ has rank 1. Thus, $ee^T - \sum_{l \neq i,j} \vec{X_l} \vec{Y_l}^T$ is the sum of $n-1$ matrices of rank 1. We know from linear algebra that the rank of sum matrices is at most of the sum of the ranks of the individual matrices (see Section 3.5 of [2]). Thus, rank$(I_n + \vec{X_i} \vec{Y_i}^T + \vec{X_j} \vec{Y_j}^T) =$ rank$(ee^T - \sum_{l \neq i,j} \vec{X_l} \vec{Y_l}^T) \le n-1 < n$. By the Fundamental Theorem of Invertible Matrices, an $n \times n$ matrix B is invertible iff rank$(B) = n$. Thus, by contrapositive, $I_n + \vec{X_i} \vec{Y_i}^T + \vec{X_j} \vec{Y_j}^T$ is noninvertible. Thus, also by the contrapositive of the Fundamental Theorem of Invertible Matrices, $det(I_n + \vec{X_i} \vec{Y_i}^T + \vec{X_j} \vec{Y_j}^T) = 0$.

Therefore,

$$
\begin{aligned}
0 &= det(I_n + \vec{X_i}\vec{Y_i}^T + \vec{X_j}\vec{Y_j}^T) && (6)\\
&= det\left(I_n + [\vec{X_i}\ \vec{X_j}]\begin{bmatrix}\vec{Y_i}^T\\ \vec{Y_j}^T\end{bmatrix}\right) && (7)\\
&= det\left(I_2 + \begin{bmatrix}\vec{Y_i}^T\\ \vec{Y_j}^T\end{bmatrix}[\vec{X_i}\ \vec{X_j}]\right) && (8)\\
&= det\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & \vec{Y_i}^T\vec{X_j}\\ \vec{Y_j}^T\vec{X_i} & 0\end{bmatrix}\right) && (9)\\
&= det\left(\begin{bmatrix}1 & \vec{Y_i}^T\vec{X_j}\\ \vec{Y_j}^T\vec{X_i} & 1\end{bmatrix}\right) && (10)\\
&= 1 - (\vec{Y_i}^T\vec{X_j})(\vec{Y_j}^T\vec{X_i}). && (11)
\end{aligned}
$$

Equation (8) follows from Lemma 5.5, (7) and (9) from the factorization of matrix products, and the rest from basic matrix arithmetic.

Thus, $(\vec{Y_i}^T\vec{X_j})(\vec{Y_j}^T\vec{X_i}) = 1$. Note that both $\vec{Y_i}^T\vec{X_j}$ and $\vec{Y_j}^T\vec{X_i}$ are of dimension $1 \times n * n \times 1 = 1 \times 1$, and thus are scalars. And since all the entries in these matrices are 0s and 1s, this implies that $\vec{Y_i}^T\vec{X_j} = 1$ for all $i \neq j$.

Therefore, by the definition of YX, we can conclude that

$$
\begin{aligned}
YX &= \begin{bmatrix}\vec{Y_1}^T\\ \vdots\\ \vec{Y_n}^T\end{bmatrix}[\vec{X_1}\ \ldots\ \vec{X_n}]\\
&= [\vec{Y_k}^T\vec{X_l}]\\
&= [b_{ij}], \text{ where } b_{ij} = \begin{cases}1 & \text{for } i \neq j\\ 0 & \text{otherwise}\end{cases}\\
&= J_n - I_n\\
&= XY
\end{aligned}
$$

Thus, X commutes with Y. By definition, it also commutes with X and I_n. Thus, X commutes with $YX + I_n = J_n$. A matrix commutes with J_n iff its row and column sums are all equal (since $XJ_n = \sum_{i=1}^{n}$ rows of $X * 1$ and

$J_nX = \sum_{i=1}^{n} 1 *$ columns of X). Thus, $XJ_n = J_nX = r * J_n$ for some scalar $r \in \mathbb{Z}^+$.

A parallel argument shows that $YJ_n = J_nY = s * J_n$ for some scalar $s \in \mathbb{Z}^+$.

Therefore,

$$\begin{aligned}
(rs)J_n &= (XY)J_n \\
&= (J_n - I_n)J_n \\
&= J_n^2 - J_n \\
&= nJ_n - J_n \\
&= (n-1)J_n \\
rs &= n-1
\end{aligned}$$

Thus, $XY = YX$, and there exist $r, s \in \mathbb{Z}^+$ with $rs = n-1$ such that $XJ_n = J_nX = rJ_n$ and $YJ_n = J_nY = sJ_n$. ∎

Now we have all the ingredients for the proof of the Theorem 4.1, restated in graph theoretical terms below:

Theorem 5.7 (The Divorce Theorem)**.** For $n \geq 2$, let $M = \{m_1, \ldots, m_n\}$, $W = \{w_1, \ldots, w_n\}$, and $E = \{m_iw_j \in M \times W | i \neq j\}$. If $B = B(M, W, E)$ is a bipartite graph, then $bp(B) = n$.

Furthermore, if $(X_1, Y_1), \ldots, (X_n, Y_n)$ is an exact biclique partition of B, then there exist $r, s \in \mathbb{Z}^+$ such that

a) $rs = n - 1$

b) $|X_i| = r$ and $|Y_i| = s$ for each $i = 1, \ldots, n$.

c) Each element of M is in exactly r of the X_i and each element of W is in exactly s of the Y_i.

d) For $i \neq j$, there exists exactly one $1 \leq k \leq n$ such that $m_kw_k \in X_i \times Y_j$.

Proof. Let $(X_1, Y_1), \ldots, (X_p, Y_p)$ be a biclique partition of B. By Lemma 5.4, $bp(B) = n$, proving the first part of the theorem.

Let X be the $n \times p$ matrix whose i^{th} column is $\vec{X_i}$, and let Y be the $p \times n$ matrix whose i^{th} row is $\vec{Y_i}^T$. By Lemma 5.4, $XY = A = J_n - I_n$. Because $p = n$ (making $(X_1, Y_1), \ldots, (X_p, Y_p)$ an exact biclique partition), Lemma 5.6 applies and there exists $r, s \in \mathbb{Z}^+$ such that $rJ_n = XJ_n$, $sJ_n = YJ_n$, and

$rs = n - 1$, where $r = |X_i|$ and $s = |Y_i|$. Thus, we have proven Parts a) and a) of the second part.

Now, also pulling from Lemma 5.6, we know that $XJ_n = rJ_n = R_n = [r_{kl}]$, where $r_{kl} = r$ for all $k, l \leq n$. Thus, $XJ_n = [r_{kl}]$, and so

$$\begin{aligned} r_{kl} &= X^k J^l \text{ (where } A^k = k^{th} \text{ row of } A) \\ &= \sum_{i=1}^{p} x_{ki} j_{il} \\ &= \sum_{i=1}^{p} x_{ki} * 1 \\ &= \sum_{i=1}^{p} x_{ki} \\ &= r. \end{aligned}$$

Thus, $\sum_{i=1}^{p} x_{ki} = r$. And since every x_{ki} is either 0 or 1, it follows that given any $m_k \in M$, m_k is an element of exactly r of the X_is.

Furthermore, Lemma 5.6 tells us that $J_n Y = sJ_n = S_n = [s_{kl}]$, where where $s_{kl} = s$ for all $k, l \leq n$. A parallel argument will show that given any $w_l \in W$, w_l is an element of exactly s of the Y_is. Therefore, we have proven Part c) of the second part.

Finally, consider the matrix YX. By Lemma 5.6, we know that $YX = XY$. Thus,

$$\begin{aligned} A &= YX \\ &= \begin{bmatrix} \vec{Y_1}^T \\ \vdots \\ \vec{Y_n}^T \end{bmatrix} [\vec{X_1} \ \cdots \ \vec{X_n}] \\ &= J_n - I_n \end{aligned}$$

Notice, then, that for any entry of $A = [a_{ij}]$,

$$\begin{aligned} a_{ij} &= i^{th} \text{ row } \cdot j^{th} \text{ column} \\ &= \vec{Y_i}^T \cdot \vec{X_j} \\ &= 1 \text{ for } i \neq j. \end{aligned}$$

Since $\vec{Y_i}^T \cdot \vec{X_j} = \sum_{k=1}^{n} y_{ik}x_{kj}$, this implies that $\sum_{k=1}^{n} y_{ik}x_{kj} = 1$ for $i \neq j$.

Now, since we are only dealing with $(0,1)$-matrices, every $y_{ik}x_{kj}$ can be only 0 or 1. Therefore, we know that $\sum_{k=1}^{n} y_{ik}x_{kj} = 1$ if and only if $y_{ik}x_{kj} = 1$ for exactly one $1 \leq k \leq n$. Furthermore, this will only occur when $y_{ik} = x_{kj} = 1$. Therefore, by the definition of $\vec{Y_i}^T$ and $\vec{X_j}$, $m_k \in X_j$ and $w_k \in Y_i$ for exactly one $1 \leq k \leq n$ for each $i \neq j$. Recalling our interpretation of the vectors from before, this means that given any two parties, the union of the men from one set and the women from the other will contain exactly one estranged couple. Therefore, we have proven Part d) of the second part, and we have proven Theorem 5.7. ∎

There are a couple of interesting consequences of the Divorce Theorem. First, if $n = p + 1$ for some prime p, notice that $n - 1$ is prime. Therefore, the only two possible values of r and s are 1 and $n-1$, and so only two exact biclique partitions are possible: the ones that correspond to the *Harem* and *Belle of the Ball* arrangements from the original example. On the other hand, if $n - 1$ is composite, other arrangements may possible (as with the Double-Blind Date example). But whatever the value of n, the Yenta will end up paying for exactly n parties when minimizing her cost.

6 Conclusion

Now, at last, the mathematical journey of the Wise Yenta comes to an end. We have followed her tale both through managing marriage and dealing with divorce, using graph theory to help bring happiness to others. But the story does not really end at the altar. The mathematics we have examined is actually quite versatile and helpful in many other disciplines. For instance, not only did the Yenta herself originate in a physics article, but the Marriage Theorem has been used for years in such applications as matching doctors to residency positions. More recently, the 2012 Nobel Prize was recently awarded to a pair of economists working on a variation of the Marriage Problem, incorporating the idea of match stability–that is, how to find a match most likely to succeed given the individual preferences of the people involved.

Furthermore, notice that while both of the Wise Yenta Theorems were ostensibly graph theory problems, each of the proofs made use of a different mathematical technique or discipline. In fact, the very idea of mathematical 'disciplines' is misleading: if nothing else, this paper shows us that mathematics is not a patchwork quilt. It is a richly embroidered tapestry, and knowledge of other mathematics can be extremely useful in solving the current problems.

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