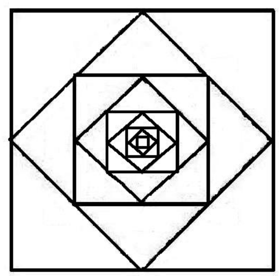
The Four-Numbers Game



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1. Introduction to the Four Numbers Game

The Four-Numbers game is a simple yet interesting problem that illustrates the fact that an elementary game relying only on basic arithmetic can exhibit complexity worthy of advanced analysis. Although the Four-Numbers game has been examined by many different people, the earliest record of the game is credited to E. Ducci of Italy in the late nineteenth century. Hence it is sometimes called a Ducci Sequence. [6] The rules of the Four-Numbers game are very simple. In fact, the game can be enjoyed by elementary school math students and college students alike.

The most basic form of the game begins with four nonnegative integers, a, b, c, d and a square. One number is placed at each corner of the square. The resulting square with numbers at each vertex is called the "start square".

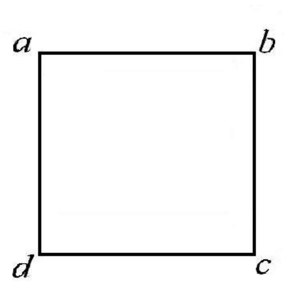

Fig 1: The initial configuration of the Four Numbers game.

Consider the start square as 'step 0' or S_0 of the game S. The first step is then obtained by creating a new square inside the original square with corners at the midpoints of the sides of the previous square. Each new corner is labeled with the absolute value of the difference between the two neighboring labels. This process continues until the difference between the numbers at each of the vertices is zero. The game is over when all four labels are zeros.

Definition: 1.1 If for some Four Numbers game $S = S_o = (a, b, c, d)$ and $a = b = c = d = 0$, then S is called the ***zero game***. All other games are ***nonzero***.

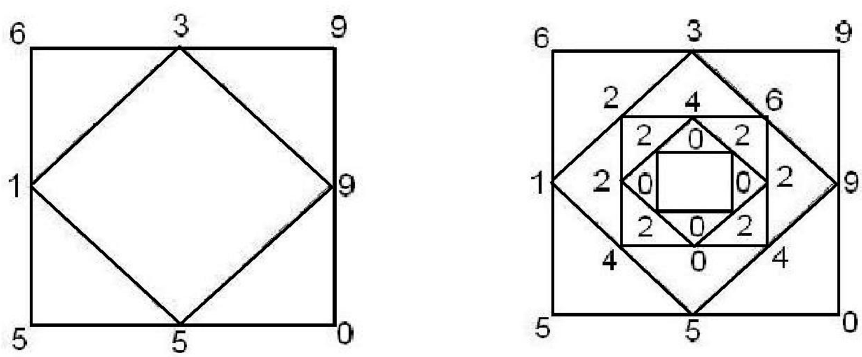

Fig. 2: *(Left)*: The $(6, 9, 0, 5)$-game after 1 step.
(Right): The $(6, 9, 0, 5)$-game ends after 5 steps.

Definition: 1.2 The ***length*** of the game $S = S_0 = (a, b, c, d)$ is defined as the number of steps, n, it takes to end the game. The length is denoted $L(a, b, c, d) = L(S_0) = n$.

For example, the $(6, 9, 0, 5)$-game has length 5, and the $(12, 4, 16, 8)$-game has length 3. See Figures 2 and 3.

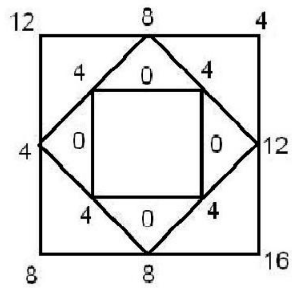

Fig. 3: The $(12, 4, 16, 8)$-game has length 3.

Each of the examples we have considered have ended in finitely many steps. Such games are said to have finite length. If a game does not end in finitely many steps, then it is said to have infinite length.

An interesting observation is that the length of a game is not necessarily dependent on the size of the numbers involved. For example the $(11, 9, 7, 3)$-game has length 7 but the $(17231205, 61305, 5371311, 322)$-game only has length 4. At first glance it would be easy to jump to the conclusion that the second game would be longer due to the larger numbers and the fact that they are further apart from one another.

The Four Numbers game is complex, and we will need to examine its behavior carefully to understand the possible long term outcomes of the game. Will will start by considering the symmetry inherent to the game.

2. Symmetries of a Square

In general, we use the word symmetry when describing objects, regions or patterns that exhibit similarity in size, shape, color, etc. In art, symmetry describes something that is visually appealing under certain guidelines. With respect to math, we generally describe a figure or expression as symmetric if its elements or parts can be interchanged in some logical way without affecting the overall object.

In order to apply symmetry to the Four Numbers game consider the symmetry of a square. (This section on the symmetries of a square follows the presentation provided by Gallian [3]). Imagine a glass square with corners labeled $1, 2, 3, 4$. Suppose we pick the glass square up off of a surface and move it around in some way before placing it back down. We are not able to change the position of labeled corners, but we can move the square in such a way that changes the order in which we see the labels. Consider rotation. If we pick up the square and rotate it 90^o in the clockwise direction we have maintained the order of the labels but their locations have changed. We can rotate the square 90^o, 180^o, or 270^o before repeating the original positions of the labels, see Figure 4.

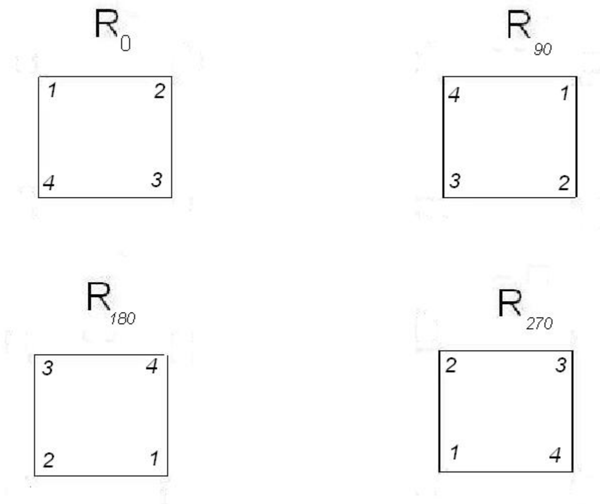

Fig. 4: All of the possible rotations of the square.

We can also consider reflecting the square about the axes of the square. That is, we can pick up the square and rotate in three-dimensional space, as opposed to the two-dimensional rotations seen in Figure 4. These three-dimensional rotations about the horizonal, vertical or one of the two diagonal axes result in the reflection of the labels of the corners. See Figure 5 for a clear visual representation of the reflections of the square.

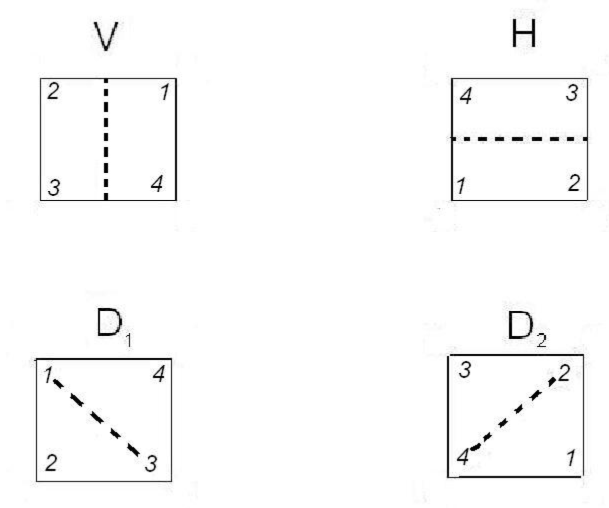

Fig. 5: All of the possible reflections of the square.

We do not necessarily have to pick up the square and move it in only one way before setting it back down. We are able to move the square in more one way followed by a move in another way. In fact, applying two such moves on the square defines a "multiplication" on the set of symmetries of the square. This multiplication is an important notion in group theory because the set of symmetries of the square together with the multiplication exhibit a nice structure known as a dihedral group. The set of the eight symmetries of the square is known as the dihedral group of order 8, or D_8.[3]

The notion of a dihedral group can be generalized to describe the symmetries on any regular polygon having n sides. As it turns out, dihedral groups are a staple of group theory and much can be said about them. However, in this paper we will not need more than what has already been presented.

To illustrate an interesting and surprising observation consider the (a, b, c, d)-game. Reflect it over D_2 then rotate it by R_{180}.

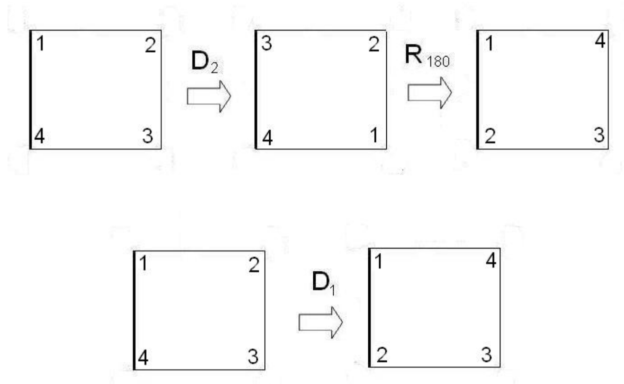

Fig. 6: *Top:* The (a, b, c, d)-game reflected over D_2 and rotated by R_{180}.
*Bottom:*The (a, b, c, d)-game reflected over D_1.

Again start with the (a, b, c, d)-game. Rotate it by R_{90}, reflect it over V then over H.

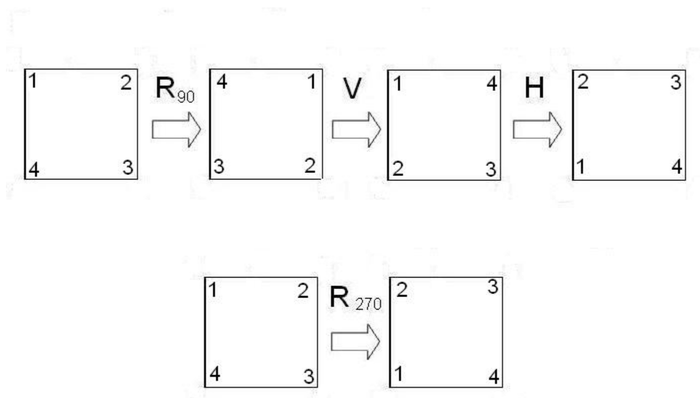

Fig. 7: *Top:* The (a, b, c, d)-game rotated by R_{90}, reflected over V then over H.
*Bottom:*The (a, b, c, d)-game rotated by R_{270}.

These examples should motivate the observation that the resulting start square of any product of the 8 symmetries of D_8 can be obtained from any one single rotation or reflection.

3. Symmetry of the Four Numbers Game

Since The Four Numbers game is defined on a square, we can use the symmetries of the square described in Section 2 to better understand the Four Numbers game and its symmetries. To begin, consider the (a, b, c, d)-game and it's 4 rotations: the (a, b, c, d)-game, the (d, a, b, c)-game, the (c, d, a, b)-game and the (b, c, d, a)-game. It is not difficult to see that these 4 games are really the same game. All of the steps of the games are all exactly the same, just with the labels in rotated positions. Thus it is no surprise that these 4 games have the same length. More generally, we can apply any of the eight symmetries of D_8 to the (a, b, c, d)-game (see Figure 6) without changing the length of the game.

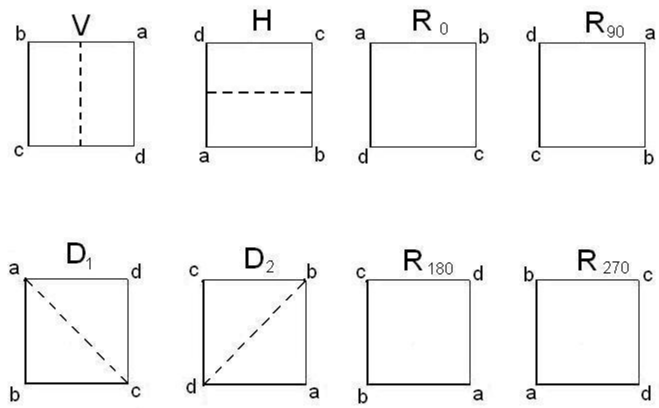

Fig.8: All of the possible rotations and reflections of the (a, b, c, d)-game under D_8.

These observations about the symmetry within the Four Numbers game should motivate the following definition.

Definition: 3.1 A Four Numbers game is ***equivalent*** to the (a, b, c, d)-game if it can be obtained from the (a, b, c, d)-game through rotations and/or reflections. Therefore all combinations in D_8 are equivalent.

We know that for any (a, b, c, d)-game, we can rotate or reflect the start square with any combination of the symmetries of D_8 without affecting the length of the game. Thus all equivalent games have the same length.

Furthermore, we can see that, in all of the symmetries presented in Figure 6, the labels a and b are next to each other. The same can be said for b and c, c and d, and d and a.

Definition: 3.2 Any of these pairs of labels that are next to each other in a start square, and thus in all possible rotations or reflections of that start square, are called ***next neighbors***.

It is clear to see that the symmetries of the square preserve next neighbors.

4. The Four Numbers Game Beyond the Integers

Thus far we have only seen integer-valued Four Numbers games[1]. Although this simple form of the game is fun and interesting, the curious reader will question what happens to the Four Numbers game if non-integer labels are used.

Consider a rational-valued Four Numbers game.

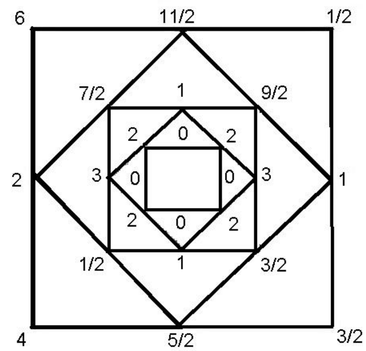

Fig. 9: The $(6, 1/2, 3/2, 4)$-game has length 5.

The rational-valued Four Numbers game is played with the same rules as the integer-valued game. In fact, the rational-valued Four Numbers game exhibits many of the same properties, as we will demonstrate in the next section, as the integer-valued Four Numbers game.

While the added complexity of rational-valued Four Numbers game may satisfy the aforementioned curious reader, an even more inquisitive reader may question the what happens to the Four Numbers game if irrational, real-valued labels are used. In moving to the rationals it will be helpful to introduce a new formulation of the game. We will no longer be only using squares to illustrate the steps of the Four Numbers game, rather we will work with quadruples in $\mathbf{R}^4$. Letting $S_0 = (a_0, b_o, c_o, d_o)$ represent the start square of a game and $S_i = (a_i, b_i, c_i, d_i)$, the i^{th}*step*, we see then that

$$S_i = (|a_{i-1} - d_{i-1}|, |a_{i-1} - b_{i-1}|, |b_{i-1} - c_{i-1}|, |c_{i-1} - d_{i-1}|).$$

Let's consider the $(\sqrt{2}, e, \pi, \sqrt{3})$-game.

[1]In actuality, we only considered nonnegative integer-valued Four Numbers games. However, because the iteration rule of the Four Numbers game is defined using the absolute value of differences, negative integer games quickly become positive integer games. Therefore, although negative-valued games are interesting, they are not worthy of additional analysis.

$S_0 = (\sqrt{2}, e, \pi, \sqrt{3})$
$S_1 = (|\sqrt{2} - e|, |e - \pi|, |\pi - \sqrt{3}|, |\sqrt{3} - \sqrt{2}|)$
$S_2 = (||\sqrt{2} - e| - |e - \pi||, ||e - \pi| - |\pi - \sqrt{3}||, ||\pi - \sqrt{3}| - |\sqrt{3} - \sqrt{2}||, ||\sqrt{3} - \sqrt{2}| - |\sqrt{2} - e||)$
$S_3 = (|||\sqrt{2} - e| - |e - \pi|| - ||e - \pi| - |\pi - \sqrt{3}|||, |||e - \pi| - |\pi - \sqrt{3}|| - ||\pi - \sqrt{3}| - |\sqrt{3} - \sqrt{2}|||,$
$|||\pi - \sqrt{3}| - |\sqrt{3} - \sqrt{2}|| - ||\sqrt{3} - \sqrt{2}| - |\sqrt{2} - e|||, |||\sqrt{3} - \sqrt{2}| - |\sqrt{2} - e||-$
$||\sqrt{2} - e| - |e - \pi|||)$
$S_4 = (0, 0, 0, 0)$

Surprisingly after only 4 steps the entries of the $(\sqrt{2}, e, \pi, \sqrt{3})$-game cancel themselves out! We will continue our discussion of real-valued Four Numbers games in Sections 7 and 8.

5. Four Numbers Games of Finite Length

We claim that every Four Numbers game with nonnegative rational number labels has finite length. Furthermore, a maximum value for the length of any game S can be calculated from the values of the labels of S_0. In order to prove this fact we will need the following observations and lemma. (Observations 1 and 2 and Lemma 1 follow from the examples and proof outlined in Sally[5].)

Observation 5.1: Multiplication of the four nonnegative rational start numbers of a game by a positive integer m does not change the length of the game.

To illustrate consider the $(6, 4, 9, 8)$-game and $(18, 12, 27, 24)$-game. Both have length 5. More generally, if $m \in \mathbf{Z}^+$ then it is not hard to see that the (a, b, c, d)-game and the (ma, mb, mc, md)-game have the same length. Consider the entries of the k^{th} step of the (ma, mb, mc, md)-game. They are equal to m times the entries of the k^{th} step of the (a, b, c, d)-game. If the (a, b, c, d)-game has length L then all of the entries in step L are equal to zero. It follows that in step 0 through step $L-1$ at least one entry is nonzero. Since m is a nonnegative integer we know that for some integer n, in order for $m \cdot n = 0$, n would have to be equal to zero. Thus if at least one entry in step $L-1$ is nonzero then at least one of m times that entry is still nonzero. Thus the (ma, mb, mc, md)-game does not end in the first $L-1$ steps. Since the (a, b, c, d)-game has length L, we know that all of the entries in step L are equal to zero. Since $m \cdot 0 = 0$ we know that all the entries of the (ma, mb, mc, md)-game in step L are also equal to zero. Therefore the (ma, mb, mc, md)-game has length L as well.

Observation 5.2: If a Four Numbers game with nonnegative integer start numbers has length at least 4, then all the numbers appearing from step 4 onward are even.

To verify observation 2 we must consider all possible cases of different combinations of even and odd numbers as the labels of a start square. It will be helpful to introduce a new labeling system. If a label for the start square is an even number, replace it with the letter 'e'. If the label is an odd number, replace it with an 'o'. For example, the $(6, 9, 4, 7)$-game becomes the (e, o, e, o)-game. Since there are two possible labels for each of the four corners of the square, there are $2^4 = 16$ possible starting configurations using this new labeling system. However, some of these starting configurations are equivalent games under reflection and rotation. In particular, we can use rotations and reflections to obtain all 16 possible starting configurations from the following 6 cases of start games:

$$\begin{aligned} (i) &= (e, e, e, e) & \qquad (iv) &= (e, o, e, o) \\ (ii) &= (e, e, e, o) & (v) &= (e, o, o, o) \\ (iii) &= (e, e, o, o) & (vi) &= (o, o, o, o) \end{aligned}$$

The following rules will be helpful:

$$\begin{aligned} e - e &= e & \qquad o - o &= e \\ e - o &= o & o - e &= o \end{aligned}$$

Figures 10-12 below show that for all 6 case start games all of the labels are even in 4 or fewer steps. Therefore, with use of rotation and reflection, all 16 possible starting configurations will have all even labels in 4 or fewer steps.

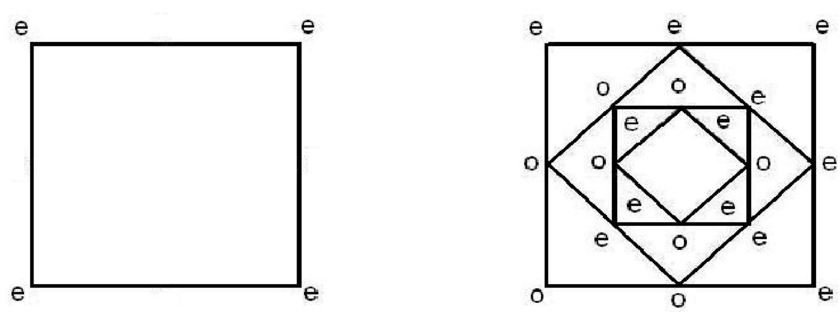

Fig 10: *(Left)*: Case (i): All labels are even after zero steps.
(Right): Case (ii): All labels are even after four steps.

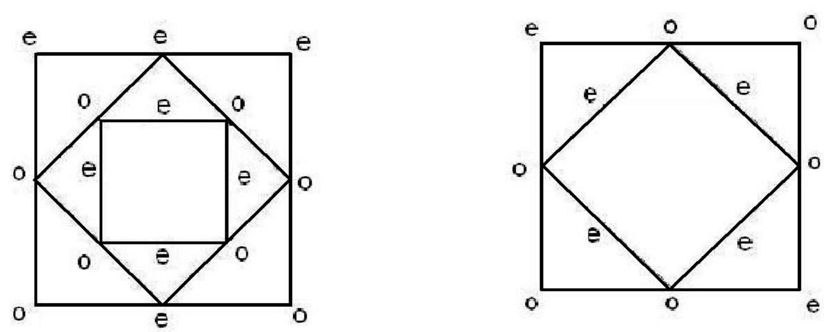

Fig 11: *(Left)*: Case (iii): All labels are even after three steps.
(Right): Case (iv): All labels are even after two steps.

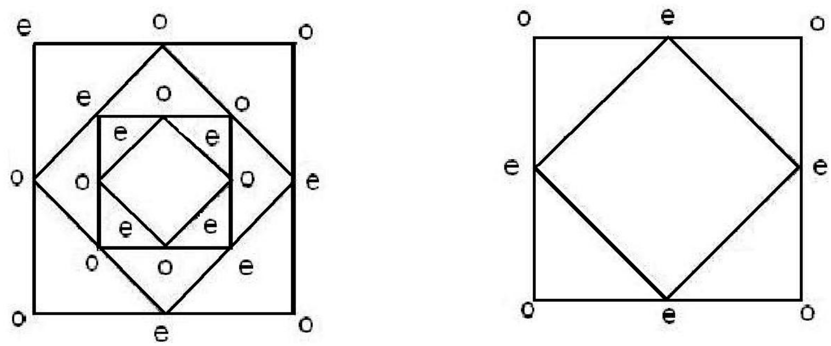

Fig 12: *(Left)*: Case (v): All labels are even after four steps.
(Right): Case (vi): All labels are even after one step.

Hence, Observation 2 holds true.

Observations 1 and 2 are helpful in proving that all integer-valued Four Numbers games end in infinitely many steps. In fact, we can even provide an upper bound on the length.

Lemma 5.1: Every Four Numbers game played with nonnegative integers has finite length. In fact, if we let A be the largest of the four nonnegative start integers and if k is the least integer such that $A/2^k < 1$, then the game has length of most $4k$.

Proof: We consider two cases: if the length of the game is at most 4 and the length of the game is greater than 4.

Case 1: Let A be the largest entry of the start square and assume the length of a game is at most 4. If $A = 0$ then all entries must be zero and $A < 1 = 2^0$. We have defined k as the least integer such that $A/2^k < 1$ that is $A < 2^k$, so in this case $k = 0$. We know the length of the zero game (the $(0,0,0,0)$-game) is $0 = 4 \cdot 0$. If $A > 0$ then k cannot equal zero, so $k \geq 1$, and certainly $4 \leq 4k$ when $k \geq 1$. We have assumed the length of the game is at most 4 and $4k = 4$ if and only if $k = 1$.

Case 2: Consider a game with length t, where $t > 4$. We know all entries in steps 4 through t are even by Observation 2. Let A equal the largest of the four integer entries in step 4. Because all of the entries are even, we can divide them by 2 without changing the length of the game by Observation 1. Thus the largest integer in this new game will be $A/2$. If the length of the new game is s, then $s + 4 = t$. If $s > 4$ then (by applying observation 2 again) we know that all the entries in steps 4 through s of the new game are even integers. Thus we can see that if $s > 4$ then $t > 8$ and all the entries in steps 8 through t of the original game are divisible by $4 = 2^2$. We can repeat this process by creating another game by dividing the entries of the 4^{th} step of the new game by 2. This yields a game with largest integer $A/2^2$ and with length q, where $q + 8 = t$. Hence we can say that the original game has length greater than or equal to 8, and the integer entries of steps 8 through t can be divided by 2^2. So if A is the largest integer in step 8 of the original game we know that $A/2^2 \in \mathbf{Z}^+$. So $A/2^2 \geq 1$. If 2 is the largest integer J such that $A/2^J \geq 1$ then $A/2^{J+1} < 1$. So in this case, $J = 2$, $J + 1 = 3$ and $A < 2^3$. That is, $k = 3$ is the smallest integer satisfying $A < 2^k$. We note now that the length of the game is at most 12. Indeed, if the length of the original game is greater than 12 then the process can be repeated again such that A can be divided by 2^3. We have said that J is the largest integer such that $A/2^J \geq 1$ we know $J = 2$, so $J \neq 3$. Hence the length of the game is not greater than 12. Therefore we have $8 \leq t \leq 12$. We have defined $k = 3$ to be the least integer such that $A/2^k$ and $8 \leq t \leq 12 = 4 \cdot 3 = 4k$. Therefore the length of the game is at most $4k$. □

We are now ready to prove the main result of this section.

Theorem 5.1

Every Four Numbers game played with nonnegative rational numbers has finite length. In fact, if N is the largest numerator occurring when the four start numbers are written with a common denominator, then the length is at most $4k$, where $N < 2^k$.

Proof:

We know that every rational number can be expressed as some fraction, a/b, where $a, b \in \mathbf{Z}$ and $b \neq 0$. Consider the (e, f, g, h)-game, where $e, f, g, h \in \mathbf{Q}^+$. We can express each of these positive rational numbers as a fraction of two positive integers. We can also manipulate the fractions so that they are expressed with a common denominator, J, where $J \in \mathbf{Z}^+$. Let $N \in \mathbf{Z}^+$ be the largest numerator occurring when the four start numbers are written with a common denominator J. Then N/J is the largest of the four rational start numbers. Let the length of the original (e, f, g, h)-game be l. We know from Observation 1 that we can multiply the four start numbers of a game by an integer and not change the length of the game. So if we multiply each of e, f, g and h by J then the length of the (Je, Jf, Jg, Jh)-game is still l, and $Je, Jf, Jg, Jh \in \mathbf{Z}^+$. Also $N = max(Je, Jf, Jg, Jh)$. From Lemma 1 we know if N is the largest of the four nonnegative integer start numbers and k is the least integer such that $N/2^k < 1$, then the length of the game is at most $4k$, so $l \leq 4k$. □

We can conclude that any Four Numbers game with nonnegative integer or rational start labels has finite length. Next consider the Four Numbers game with long but finite length.

6. Four Numbers Games of Long but Finite Length

We have shown that all Four Numbers games with nonnegative or rational start labels have finite length. And we can see from our examples in Figures 2 and 3 that the $(6, 9, 0, 5)$-game has length 5 and the $(12, 4, 16, 8)$-game has length 3; both games have relatively short length. In fact, the longest game we have discussed is the $(11, 9, 7, 3)$-game which has length 7. This begs the question: Are there long Four Numbers games, and if yes, how do we construct them?

In order to explore Four Numbers games of long length we will consider the Tribonacci games. Following the argument in Sally [5] we define a Tribonacci game as a Four Numbers game played with four successive terms of the Tribonacci sequence. Where the Tribonacci sequence is defined recursively much like the Fibonacci sequence. In particular, by definition, $t_0 = 0, t_1 = 1, t_2 = 1$ and for all $n \geq 3, t_n = t_{n-1} + t_{n-2} + t_{n-3}$.

The length of a Tribonacci game becomes longer as the labels of the game get further out in the Tribonacci sequence. We will show that we can compute the exact length of any

Tribonacci game and that we can produce long games, but first we must familiarize ourselves with a few new concepts.

First, let us introduce some notation to simplify the upcoming computations. Let T_n be a the $(t_n, t_{n-1}, t_{n-2}, t_{n-3})$-Tribonacci game. Then

$$DT_n = (t_n - t_{n-3}, t_n - t_{n-1}, t_{n-1} - t_{n-2}, t_{n-2} - t_{n-3}),$$

and the k^{th} step of the T_n game is denoted $D^k T_n$.

Next, recall the greatest integer function, denoted $[r]$ for any real number r, is the the greatest integer less than or equal to r [5]. For Example:

$$[2.6] = 2 \qquad \left[\frac{7}{2}\right] = 3 \qquad [.333333] = 0.$$

Before we are able to prove the formula for computing the length of a Tribonacci game we need the following Lemma.

Lemma 6.1: The Four Numbers game that starts with the third step of the T_n-game has the same length as the T_{n-2}-game. That is

$$D^3 T_n = 2T_{n-2}{}^2.$$

Proof:

Using the operator D we can compute

$$\begin{aligned} D^3 T_n = (&t_n - t_{n-1} + t_{n-2} - t_{n-3}, t_n - t_{n-1} - t_{n-2} + t_{n-3}, \\ &- t_n + 3t_{n-1} - t_{n-2} - t_{n-3}, t_n - 3t_{n-1} + 3t_{n-2} - t_{n-3}) \end{aligned}$$

and using Observation 5.1 we see that $2T_{n-2} = (2t_{n-2}, 2t_{n-3}, 2t_{n-4}, 2t_{n-5})$. Solving $t_{n-1} = t_{n-2} + t_{n-3} + t_{n-4}$ for t_{n-4} yields $t_{n-4} = t_{n-1} - t_{n-2} - t_{n-3}$ and solving $t_{n-2} = t_{n-3} + t_{n-4} + t_{n-5} = t_{n-3} + (t_{n-1} - t_{n-2} - t_{n-3}) + t_{n-5}$ for t_{n-5} yields $t_{n-5} = 2t_{n-2} - t_{n-1}$. Thus using only terms of t_n, t_{n-1}, t_{n-2},and t_{n-3} we can see that

$$2T_{n-2} = (2t_{n-2}, 2t_{n-3}, 2t_{n-1} - 2t_{n-2} - 2t_{n-3}, 4t_{n-2} - 2t_{n-1}).$$

Thus we can manipulate the labels of each of the games too see:

$$\begin{aligned} t_n - t_{n-1} - t_{n-3} &= t_{n-1} + t_{n-2} + t_{n-3} - t_{n-1} - t_{n-3} + t_{n-2} \\ &= 2t_{n-2} \end{aligned}$$

$$\begin{aligned} t_n - t_{n-1} - t_{n-2} + t_{n-3} &= t_{n-1} + t_{n-2} + t_{n-3} - t_{n-1} - t_{n-2} + t_{n-3} \\ &= 2t_{n-3} \end{aligned}$$

[2] Recall from Observation 5.1 that we can multiply the labels of a Four Numbers game by an integer without affecting the length of the game. Therefore $L(T_{n-2}) = L(2T_{n-2})$.

$$
\begin{aligned}
-t_n + 3t_{n-1} - t_{n-2} - t_{n-3} &= -t_{n-1} - t_{n-2} - t_{n-3} + 3t_{n-1} - t_{n-2} - t_{n-3} \\
&= 2t_{n-1} - 2t_{n-2} - 2t_{n-3}
\end{aligned}
$$

$$
\begin{aligned}
t_n - 3t_{n-1} + 3t_{n-2} - t_{n-3} &= t_{n-1} + t_{n-2} + t_{n-3} - 3t_{n-1} + 3t_{n-2} - t_{n-3} \\
&= -2t_{n-1} - 2t_{n-2} - 2t_{n-3}
\end{aligned}
$$

Therefore $D^3 T_n = 2T_{n-2}$. □

Lemma 6.2: For all $n \in \mathbf{N}$, $\left[\frac{n-2}{2}\right] + 1 = \left[\frac{n}{2}\right]$.

Proof: Assume $\frac{n}{2} = \left[\frac{n}{2}\right] + \epsilon$, since $n \in \mathbf{N}$ we know that $\epsilon = 0$ or $\epsilon = \frac{1}{2}$. Then

$$
\begin{aligned}
\left[\frac{n}{2} - 1\right] &= \left[\left[\frac{n}{2}\right] + \epsilon - 1\right] \\
&= \begin{cases} \left[\frac{n}{2}\right] - 1 & \text{if } \epsilon = 0 \\ \left[\frac{n}{2}\right] - 1 & \text{if } \epsilon = \frac{1}{2}. \end{cases}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left[\frac{n-2}{2}\right] + 1 &= \left[\frac{n}{2} - 1\right] + 1 \\
&= \left[\frac{n}{2} - 1\right] + 1 \\
&= \left[\frac{n}{2}\right] - 1 + 1 \\
&= \left[\frac{n}{2}\right].
\end{aligned}
$$

□

Now we are ready for our next theorem regarding long games.

Theorem 6.1

Given any $n \in \mathbf{N}$, the Tribonacci Four Numbers game $T_n = (t_n, t_{n-1}, t_{n-2}, t_{n-3})$ has length $3\left[\frac{n}{2}\right]$.

Proof: We proceed by complete induction. For the base Case, $n = 3$, we know that $t_3 = t_2 + t_1 + t_0$, where $t_3 = 2 = 1 + 1 + 0$. We can easily compute the length of the $(2, 1, 1, 0)$-game:

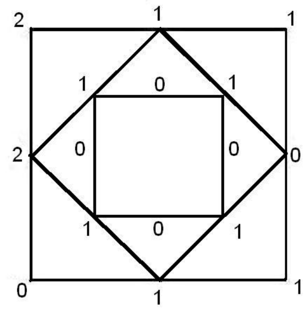

Fig. 13: $L(T_3) = 3$.

We see that $\left[\frac{3}{2}\right] = 1$. Thus $L(T_3) = 3 = 3{\cdot}1 = 3\left[\frac{3}{2}\right]$. Now assume $n \geq 4$ and $L(T_k) = 3\left[\frac{k}{2}\right]$. for all $k \in \mathbf{N}$ such that $n \leq k \leq n$ and consider $L(T_n)$. We know from Lemma 6.1 that $D^3T_n = 2T_{n-2}$, which to say that $L(T_n) = L(T_{n-2}) + 3$. Since $n \geq 4$ we know that $n-2 \geq 2$. Thus by our base case we know that $L(T_{n-2}) = 3\left[\frac{n}{2}\right]$. Therefore using Lemma 6.2 we can see that

$$
\begin{aligned}
L(T_n) = L(T_{n-2}) + 3 &= 3\left[\frac{n-2}{2}\right] + 3 \\
&= 3\left(\left[\frac{n-2}{2}\right] + 1\right) \\
&= 3\left(\left[\frac{n}{2} - \frac{2}{2}\right] + 1\right) \\
&= 3\left(\left[\frac{n}{2}\right] - [1] + 1\right) \\
&= 3\left(\left[\frac{n}{2}\right] - 1 + 1\right) \\
&= 3\left[\frac{n}{2}\right]
\end{aligned}
$$

Therefore for all $n \in \mathbf{N}, L(T_n) = 3\left[\frac{n}{2}\right]$. □

For example, consider $t_{18} = 19,513$. Then $T_{18} = (19513, 10609, 5868, 3136)$. Then we can use Theorem 6.1 to see that $L(T_{18}) = 27$. We can also use this result to produce games of arbitrarily long length. Let's construct a Four Numbers game length greater than or equal to 100. So $100 < L(T_n) = 3\left[\frac{n}{2}\right]$. Solving for n we see $n > 66.66$ but $[66] = [67]$ therefore we must add 1 to our n. Thus $L(T_{68}) > 100$. In fact $L(T_{68}) = 102$.

Thus by considering the Tribonacci game T_n for sufficiently large n we can construct Four Numbers games of arbitrarily large length.

7. The Four Numbers Game over the Real Numbers

Thus far our examination of the Four Numbers game has been relatively basic. Our analysis has relied largely on arithmetic and the fact that there is a minimal nonnegative nonzero integer. Note that there is no minimal nonnegative real number so the Four Numbers game with real valued labels is quite different from the rational or integer cases.

In moving to the reals our analysis will become more complex, and we will use some mathematical tools from linear algebra. It is natural to think of a real-valued quadruple as a vector in $\mathbf{R}^4$. Thus the iteration rule for the Four Numbers game is the function $F : \mathbf{R}^4 \to \mathbf{R}^4$ defined by $F(a, b, c, d) = (|a-d|, |a-b|, |b-c|, |c-d|)$. Can we represent F with a matrix? That is, is F a linear transformation? Recall that $F : \mathbf{R}^4 \to \mathbf{R}^4$ is a linear transformation if for all vectors $\vec{u}, \vec{v} \in \mathbf{R}^4$ and for any real number r,$F(\vec{u}+\vec{v}) = F(\vec{u})+F(\vec{v})$ and $F(r\vec{u}) = r(F(\vec{u}))$ [2].

Claim: F is not a linear transformation on $\mathbf{R}^4$.

We can verify our claim with a counterexample. Let $\vec{u}$ and $\vec{v}$ be quadruples in $\mathbf{R}^4$. Let $\vec{u} = (6, 0, 3, 1)$ and $\vec{v} = (0, 4, 1, 9)$, so that $\vec{u} + \vec{v} = (6, 4, 4, 10)$. Then

$$\begin{aligned} F(\vec{u} + \vec{v}) &= (|6 - 10|, |6 - 4|, |4 - 4|, |4 - 10|) \\ &= (4, 2, 0, 6), \end{aligned}$$

while

$$\begin{aligned} F(\vec{u}) + F(\vec{v}) &= (|6 - 1|, |6 - 0|, |0 - 3|, |3 - 1|) + (|0 - 9|, |0 - 4|, |4 - 1|, |1 - 9|) \\ &= (5, 6, 3, 2) + (9, 4, 3, 8) \\ &= (14, 10, 6, 10) \\ &\neq F(\vec{u} + \vec{v}). \end{aligned}$$

Thus F is not a linear transformation on $\mathbf{R}^4$.

The reader should not be too disappointed however, because the linearity of our function is not *completely* lost. If we restrict the map of F to the vectors in $\mathbf{R}^4$ having strictly decreasing components then we can salvage the linearity. Let S be the set $\{(a, b, c, d) \in \mathbf{R}^4 : a > b > c > d\}$. Then we can define a new function F_S such that for all $\vec{s} = (a, b, c, d) \in S$, $F_S(\vec{s}) = F_S(a, b, c, d) = (a - d, a - b, b - c, c - d)$

Claim: $F : S \to \mathbf{R}^4$ is a linear transformation.

Proof: Let $\vec{u}, \vec{v} \in S$. So $\vec{u} = (g, h, i, j)$ and $\vec{v} = (k, l, m, n)$ where $g > h > i > j$ and $k > l > m > n$. Then we can see that $\vec{u}+\vec{v} = (g+k, h+l, i+m, j+n)$, and we can compute

$F_S(\vec{u}), F_S(\vec{v})$ and $F_S(\vec{u}+\vec{v})$. Then

$$\begin{aligned} F_S(\vec{u}+\vec{v}) &= ((g+k)-(j+n),(g+k)-(h+l),(h+l)-(i+m),(i+m)-(j+n)) \\ &= (g+k-j-n, g+k-h-l, h+l-i-m, i+m-j-n) \\ &= ((g-j)+(k-n),(g-h)+(k-l),(h-i)+(l-m),(i-j)+(m-n)) \\ &= (g-j, g-h, h-i, i-j)+(k-n, k-l, l-m, m-n) \\ &= F_S(\vec{u}+\vec{v}). \end{aligned}$$

Thus the first condition of linearity is satisfied. To show that the second condition also holds let $\vec{u} \in S$. So $\vec{u} = (g,h,i,j) \in \mathbf{R}^4$ such that $g > h > i > j$. Let r be any real number. Then we can easily compute F_S for these vectors.

$$\begin{aligned} F_S(r\vec{u}) &= (rg-rj, rg-rh, rh-ri, ri-rj) \\ r(F_S(\vec{u})) &= r(g-j, g-h, h-i, i-j) \\ &= (rg-rj, rg-rh, rh-ri, ri-rj) \\ &= F_S(r\vec{u}) \end{aligned}$$

□

Thus the second condition is satisfied and $F_S : S \to \mathbf{R}^4$ is a linear transformation, hence F_S can be represented by a matrix. In fact, we see that the linear transformation $F_S : S \to \mathbf{R}^4$ is represented by the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

because

$$F_S(\vec{v}) = F_S \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-d \\ a-b \\ b-c \\ c-d \end{bmatrix}.$$

Since the transition rule of the Four Numbers game is now described as a matrix we can observe that the Four Numbers game $s_0 = (a,b,c,d) \in S$ has length n if $F_S^n(\vec{s_0}) = M^n(\vec{s_0}) = \vec{0}$. Additionally, a game $\vec{s_0}$ has infinite length if $F_S^n(\vec{s_0}) = M^n(\vec{s_0}) \neq \vec{0}$ for all $n \in \mathbf{N}$. In the next section we will use eigenvectors to construct such a game.

8. Four Numbers Games of Infinite Length

Recall that $\vec{s_0} \in S$ is an eigenvector of M for the eigenvalue λ therefore $F_S(\vec{s_0}) = \lambda(\vec{s_0}) \neq \vec{0}$. Applying F_S again we conclude that $F_S(F_S(\vec{s_0})) = \lambda^2(\vec{s_0}) \neq \vec{0}$. More generally, $F_S^n(\vec{s_0}) = \lambda^n(\vec{s_0}) \neq \vec{0}$. Hence no matter how many times F_S is applied to $\vec{s_0}$, the zero vector is unobtainable. Thus $\vec{s_0}$ corresponds to a Four Numbers game of infinite length. It remains to show that there exists such an eigenvector $\vec{s_0}$.

The process for finding eigenvectors relies on the characteristic polynomial $p(\lambda)$ of a matrix M where $p(\lambda) = det(M - \lambda I_4)$. Recall that $F_S(\vec{s_0}) = M(\vec{s_0}) = \lambda(\vec{s_0})$ has a nonzero solution if $det(M - \lambda I_4)$ equals zero.
So compute

$$det(M - \lambda I_4) = det \begin{bmatrix} 1-\lambda & 0 & 0 & -1 \\ 1 & -1-\lambda & 0 & 0 \\ 0 & 1 & -1-\lambda & 0 \\ 0 & 0 & 1 & -1-\lambda \end{bmatrix}$$

$$= 2\lambda + 2\lambda^3 + \lambda^4.$$

Thus $p(\lambda) = \lambda(\lambda^3 + 2\lambda^2 - 2)$. Setting $p(\lambda) = 0$ we can see that $\lambda = 0$ is a root. Using Maple to find the other roots we see that one of the roots is real

$$\begin{aligned} \lambda_0 &= \frac{1}{3(19+3\sqrt{(33)})^{1/3}} + \frac{4}{3(19+3\sqrt{(33)})^{1/3}} - \frac{2}{3} \\ &= 0.8392867553, \end{aligned}$$

and the other two roots are complex

$$\begin{aligned} \lambda_1 &= \frac{-1}{6(19+3\sqrt{(33)})^{1/3}} - \frac{2}{(3(19+3\sqrt{(33)})^{1/3}} - \frac{2}{3} + \frac{1}{2i\sqrt{(3)}((1/3)(19+3\sqrt{(33)})^{1/3}} \\ &\quad - \frac{4}{3(19+3\sqrt{(33)})^{1/3}} \\ &= -1.419643378 + .6062907300i \\ \lambda_2 &= \frac{-1}{6(19+3\sqrt{(33)})^{1/3}} - \frac{2}{(3(19+3\sqrt{(33)})^{1/3}} - \frac{2}{3} - \frac{1}{2i\sqrt{(3)}((1/3)(19+3\sqrt{(33)})^{1/3}} \\ &\quad - \frac{4}{3(19+3\sqrt{(33)})^{1/3}} \\ &= -1.419643378 - .6062907300i. \end{aligned}$$

Then if $\vec{s_0} = (a_0, b_0, c_0, d_0)$ is the eigenvector associated with the real eigenvalue λ_0, then

$$\begin{bmatrix} 1-\lambda_0 & 0 & 0 & -1 \\ 1 & -1-\lambda_0 & 0 & 0 \\ 0 & 1 & -1-\lambda_0 & 0 \\ 0 & 0 & 1 & -1-\lambda_0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Multiplying then yields

$$\begin{bmatrix} a_0(1-\lambda_0) - d_0 \\ b_0(-1-\lambda_0) + a_0 \\ c_0(-1-\lambda_0) + b_0 \\ d_0(-1-\lambda_0) + c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\begin{aligned}(1-\lambda_0)a_0 &= d_0\\ (1+\lambda_0)b_0 &= a_0\\ (1+\lambda_0)c_0 &= b_0\\ (1+\lambda_0)d_0 &= c_0.\end{aligned}$$

Letting $d_0 = 1$ we can solve the equations for a_0, b_0 and c_0 simultaneously resulting in

$$\vec{s_0} = ((1+\lambda_0)^3, (1+\lambda_0)^2, (1+\lambda_0), 1).$$

Noting that $0 < \lambda_0 < 1$ we know $(1+\lambda_0)^3 > (1+\lambda_0)^2 > (1+\lambda_0) > 1$. Thus $\vec{s_0}$ is indeed in S, thus the linear transformation F_S holds for $\vec{s_0}$.

This series of computations (which follow the argument in Sally [5]) shows that the matrix M has an eigenvector $\vec{s_0}$ and therefore there exists a real-valued Four Numbers game of infinite length. In fact, as the next theorem states, there exists an infinite set of Four Numbers games with infinite length.

Theorem 8.1

There exists infinitely many Four Numbers games of infinite length.

Proof: We have shown that the Four Numbers game with entries $(a_0, b_0, c_0, d_0) = \vec{s_0}$, where $\vec{s_0}$ is the eigenvector corresponding to the eigenvalue λ_0, has infinite length. Let $r \in \mathbf{R}^+$ such that $r > 0$. Consider the vector $\vec{r} \in \mathbf{R}^4$ where $\vec{r} = (r, r, r, r)$. Then $\vec{s_0} + \vec{r} = (a_0 + r, b_0 + r, c_0 + r, d_0 + r)$.

We can apply our function $F_S : S \to \mathbf{R}^4$ because we know that $\vec{s_0} \in S$. So $a_0 > b_0 > c_0 > d_0$. Since $r > 0$ then $a_0 + r > b_0 + r > c_0 + r > d_0 + r$. Compute:

$$\begin{aligned}F_S(\vec{s_0} + \vec{r}) &= ((a_0 + r) - (d_0 + r), (a_0 + r) - (b_0 + r), (b_0 + r) - (c_0 + r), (c_0 + r) - (d_0 + r))\\ &= (a_0 + r - d_0 - r, a_0 + r - b_0 - r, b_0 + r - c_0 - r, c_0 + r - d_0 - r)\\ &= (a_0 - d_0, a_0 - b_0, b_0 - c_0, c_0 - d_0)\\ &= F_S(\vec{s_0})\end{aligned}$$

We know that $F_S(\vec{s_0}) \neq \vec{0}$ and $F_S^n(\vec{s_0}) \neq \vec{0}$ and we know there exists an eigenvalue λ_0 that corresponds to $\vec{s_0}$ having infinite length. Thus for all $r > 0$, $\vec{s_0} + \vec{r}$ represents a different Four Numbers game with infinite length.

We know from Observation 1 that for all $m \in \mathbf{Z}^+, L(ma, mb, mc, md) = L(a, b, c, d)$. A parallel argument will show that $L(ka, kb, kc, kd) = L(a, b, c, d)$ for all $k \in \mathbf{R}$. Hence $\vec{s_0}$ and $k\vec{s_0} + \vec{r}$ will have the same length for all $k, r \in \mathbf{R}^+$. $\square$

Now we know that not only does there exist an infinitely long Four Numbers game over there reals, but there are infinitely many of them! Thus we have shown that there are games of finite bounded length, long but finite length and infinite length.

9. The Three-Numbers Game? Five? Six?

While the Four Numbers game is very interesting, the question of the possibility of a Three Numbers game arises. Consider the $(1, 1, 2)$-game:

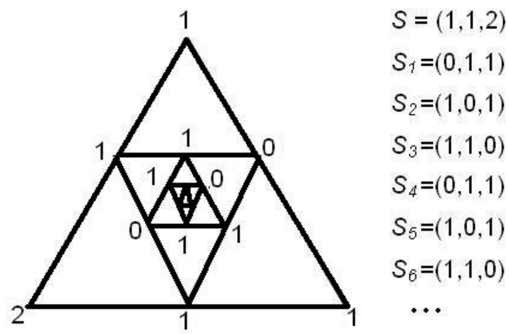

Fig. 14: The $(1, 1, 2)$-game and the first 6 steps written out.

As it turns out, this Three Numbers game does not have finite length. In fact, all Three Numbers games have infinite length unless the three start numbers are equal. A Three Numbers game that does not have three equal start numbers will begin to oscillate between 1,1, and 0, after a certain amount of steps. This will obviously never end in a $(0, 0, 0)$ step. Thus there are three possible lengths for any Three Numbers game, 0, 1, or ∞.

The Five Numbers game follows a similar pattern. Unless all five entries are the same, the Five Numbers game will have infinite length.

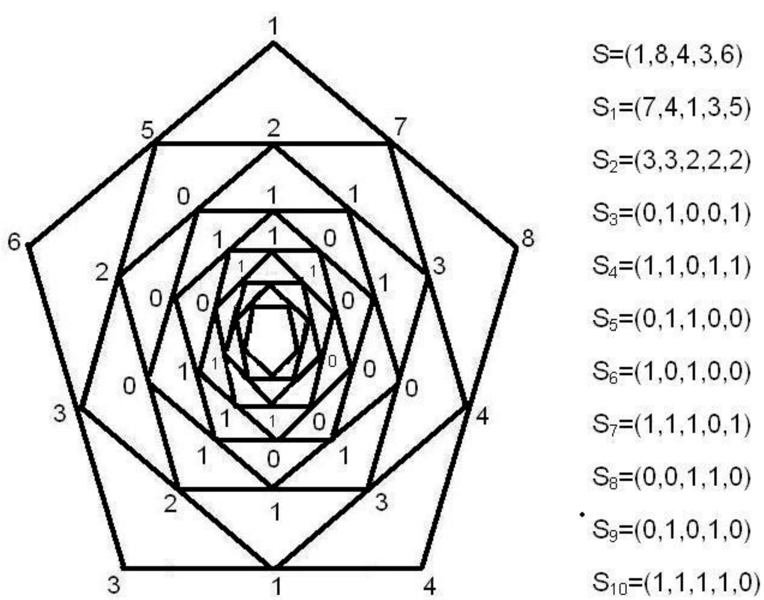

Fig. 15: The first 7 steps of the $(1, 8, 4, 3, 6)$-game and the first 10 steps written out.

One can experiment and conclude that the Six and Seven Numbers games do not end in finite length either, unless all of the entries are equal. Does such behavior continue for all K-Numbers games with $K > 7$? Let's examine the Eight Numbers game. In particular consider the $(4, 12, 6, 3, 0, 9, 1, 2)$-game.

$$\begin{aligned}
S_0 &= (4, 12, 6, 3, 0, 9, 1, 2)\\
S_1 &= (8, 6, 3, 3, 9, 8, 1, 2)\\
S_2 &= (2, 3, 0, 6, 1, 7, 1, 7)\\
S_3 &= (1, 3, 6, 5, 6, 6, 6, 5)\\
S_4 &= (2, 3, 1, 1, 0, 0, 1, 4)\\
S_5 &= (1, 2, 0, 1, 0, 1, 3, 2)\\
S_6 &= (1, 2, 1, 1, 1, 2, 1, 1)\\
S_7 &= (1, 1, 0, 0, 1, 1, 0, 0)\\
S_8 &= (0, 1, 0, 1, 0, 1, 0, 1)\\
S_9 &= (1, 1, 1, 1, 1, 1, 1, 1)\\
S_{10} &= (0, 0, 0, 0, 0, 0, 0, 0)
\end{aligned}$$

Further experimentation will lead to more finite games. Observe that both 4 and 8 are powers of 2. In fact, it is this property of 4 and 8 that ensures the finitude of the Four and Eight Numbers games. We state without proof a theorem from Sally [5][3].

Theorem 9.1

Let k be an integer greater than 2. Every k-Numbers game has finite length if and only if k is a positive power of 2.

10. Conclusion

We have shown through our analysis of the Four Numbers game that games exist of finite length, arbitrarily long length and infinite length. Our observations and lemmas have yielded some very interesting characteristics of the Four Numbers game. In fact, there is even more to the the Four Numbers game then discussed in this paper. With further analysis we can involve statistics to determine the probability that a Four Numbers game will end in k-steps[5]. We can also, with additional research, illustrate the connection between the Four Numbers game and Pascal's triangle[3]. Over all we have exhibited through detailed examination with advanced mathematics and linear algebra that the Four Numbers game is much more than a simple game of differences, it is interesting and worthy of meticulous analysis.

[3]Proof of this involves the polynomial ring $\mathbf{Z}_2$ and is beyond the scope of this paper. Sally, Judith D., and Paul J. Sally. "The Four Numbers Problem." Roots to Research: A Vertical Development of Mathematical Problems.

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