

# Polygonal Numbers and Finite Calculus

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# 1 Introduction

Most know the Greek mathematician Diophantos for the Arithmetica, one of the earliest seminal works in algebra and, indeed, mathematics generally. Yet the Greek's reach extended beyond that field and into a branch of math that has intrigued thinkers for many centuries following – polygonal numbers. He devoted a full treatise to the subject, his Polygonal Numbers, a work that survived in partial form to our modern age. In it Diophantos explains his rule for finding a polygonal number from the side of the polygon and the number of angles, which amounts to the general form of a  $d$ -gonal number of order  $n$ ,

$$p_d(n) = \frac{(d-2)n^2 + (4-d)n}{2}.$$

Diophantos also addressed polygonal numbers in the Arithmetica, which famously led Pierre de Fermat to scribble his Last Theorem in the margin of his edition. [7]

These numbers are the subject of this paper. In it I will explain their properties, how to derive specific generation and summation formulas intuitively, show how we can derive the general generation formula systematically using finite calculus, and discuss their application to other areas.

## 2 Polygonal Number Basics

Before we can explain properties and applications of polygonal numbers we must first understand what they are exactly. At first brush, we will describe polygonal numbers as numbers that can be represented by arrangements of equally spaced points that form regular polygons. This description, while accurate, begs for elaboration, so some examples are in order.

Square numbers, perhaps the most recognizable polygonal, serve as a good illustration of the subject. Figure 1 gives numerical and geometric representations of the first five square numbers.

Figure 1: The First Five Square Numbers, Showing Gnomons

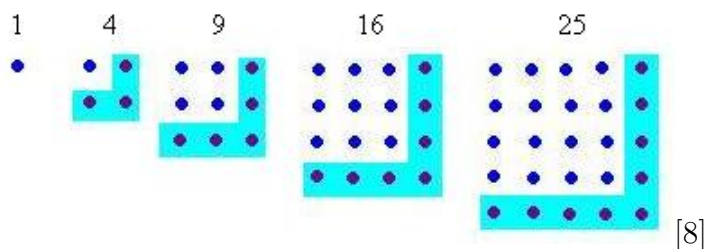


Figure 1 depicts three important characteristics about square numbers that are obvious

but important for understanding polygonal numbers generally. First, it shows, in consonance with the above description of polygonal numbers, that square numbers are able, of course, to be represented by arrangements of equally spaced points that form regular 4-gons, squares. Second, it displays the squares in a sequential way and treats them as the “first five,” implying that the numbers are ordered. Third, it highlights the difference between one square and the next in the sequence. Notice the shaded dots that make an L-shape in the second through fifth squares. These dots illustrate the “gnomon,” which is the name for the additive device that enlarges a geometric object while maintaining its shape. The name comes from the device’s shape, which in this most common case of squares resembles a sundial’s gnomon. These characteristics hold in essence for polygonal numbers generally; we will consider them for two other specific cases.

Figure 2: The First Four Triangular and Pentagonal Numbers

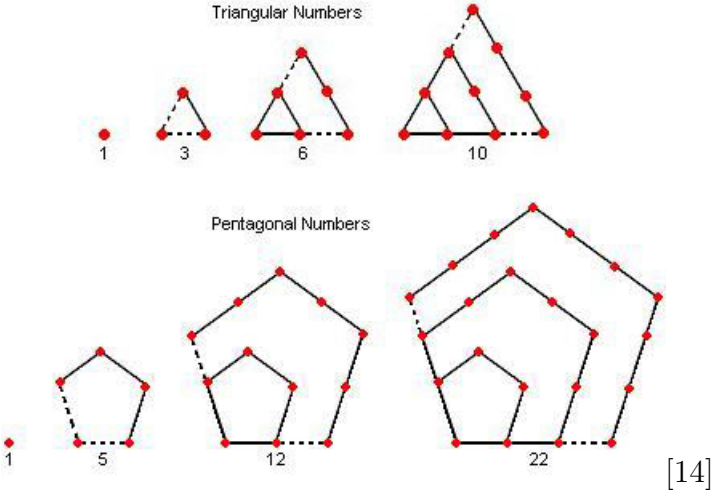


Figure 2 gives numerical and geometric representations of triangular and pentagonal, both also polygonal. As with square numbers, triangular and pentagonal numbers possess the first characteristic, the capacity to be represented by arrangements of equally spaced points that form regular 3- and 5-gons, triangles and pentagons. Also, these numbers are ordered, like the square numbers. Indeed, we refer to the place of any of these numbers as the “order” or “rank” of that number, the  $n^{th}$  number. Finally, gnomons make up the difference between consecutive triangular and pentagonal numbers – the dotted lines set them off in each representation – though they differ from each other and from those of squares.

Let us look at the gnomon a little more closely. For the triangular numbers (3-gons) we have gnomons of 1, 2, 3, and 4 dots, respectively; they increase by one as the order of triangular number increases by one. For squares we have 1, 3, 5, and 7 dots; the gnomons increase by two as the order increases. And for pentagonal numbers, we have 1, 4, 7, and 10 dots; as expected, the increase is by three as order of the 5-gons increases. It is thus evident that the difference between consecutive gnomons for a polygonal number changes when the gnomon corresponds to a different polygonal number. Specifically, the difference

increases as by one as the number of sides of the polygon depicted in the corresponding polygonal number's geometric representation increases by one.

From these examples we can surmise a formula for the gnomon,  $g_d(n)$ , that, when added to the number of  $n - 1$  rank, forms the  $d$ -gonal number of rank  $n$ ,

$$g_d(n) = 1 + (n - 1)(d - 2).$$

Note that  $n, d \in \mathbb{N}$  and  $d > 2$  (necessary in order for the corresponding geometric representation to be a polygon). Unless specified otherwise, these conditions for  $n$  and  $d$  hold throughout the rest of the paper.

Clearly, then, the simple addition of this general gnomon generates  $d$ -gonal numbers of order  $n$ , which is to say all polygonal numbers. From this recognition and the gnomon formula we arrive at the formal definition of a polygonal number.

**Definition 2.1** A “polygonal number,”  $p_d(n)$ , is a number of the form

$$p_d(n) = 1 + [1 + (d - 2)] + [1 + 2(d - 2)] + \dots + [1 + (n - 1)(d - 2)].$$

For each of the polygonal numbers we have already considered it is possible to derive a formula based on this definition. We have seen, for example, that a triangular number,  $p_3(n)$  or just  $t_n$ , is of the form

$$t_n = 1 + 2 + \dots + (n - 1) + n = \sum_{k=1}^n k.$$

This formula corresponds to the geometric representation in Figure 2. We also have seen from Figure 1 that a square number is of the form

$$s_n = 1 + 3 + 5 + \dots + (2n - 1) = \sum_{k=1}^n (2k - 1),$$

but a far more familiar way to represent squares is of course  $n^2$ , the closed-form expression. Truly, who would prefer to find the square number of a particular order by adding consecutive odd natural numbers instead of simply multiplying the order by itself? For squares of high orders this method of summing would quickly become cumbersome, and so would it be to find triangular numbers (though the sums here would be of all naturals, not simply odd ones). We would like, then, to derive a closed-form expression for triangular numbers, as  $n^2$  is for square numbers.

Finding the closed-form expression for square numbers was easy; all we had to do was to look at the geometric representation and realize that the number of dots in one side times the number in another yielded the total number of dots. Such an intuitive method

is not available to us in the case of triangular numbers. Instead, let us look to its formula as derived from the definition of polygonal numbers.

Let us write out the summands in two directions, as such:

$$t_n = 1 + 2 + \dots + (n - 1) + n$$

$$t_n = n + (n - 1) + \dots + 2 + 1.$$

If we add the two expressions, we can see that the left side will equal  $2t_n$  and each of the corresponding terms on the top and bottom of the right side will sum to  $n + 1$ . Since we have  $n$  of these terms, we can say that the right side will equal  $n(n + 1)$ . Since we wanted to find the expression for  $t_n$ , we divide both sides by two, and get  $\frac{n(n+1)}{2}$ . This process gives rise to our first theorem, which formalizes the closed-form expression for triangular numbers.

**Theorem 2.2 (Triangular Number Generating Formula)** *The  $n^{\text{th}}$  triangular number is equal to  $\frac{n(n+1)}{2}$ .*

Of the formulas for polygonal numbers this one is perhaps most famous for its discovery: young Gauss came upon it while solving a problem for a teacher. We will prove the theorem rigorously now.

**Proof:**

Let  $t_n$  be a triangular number of order  $n$ . So  $t_n = 1 + 2 + \dots + (n - 1) + n$ .

Proceed by induction on  $n$ .

Check for  $n = 1$ :  $t_1 = 1$  and  $\frac{1(1+1)}{2} = 1$ , so  $n = 1$  checks.

Assume that  $n = k$  holds true; that is,  $t_k = 1 + 2 + \dots + (k - 1) + k = \frac{k(k+1)}{2}$ .

Prove for  $n = k + 1$ :

$$\begin{aligned} t_{k+1} &= 1 + 2 + \dots + (k - 1) + k + (k + 1) \\ &= \frac{k(k + 1)}{2} + k + 1 \\ &= \frac{k(k + 1) + 2(k + 1)}{2} \\ &= \frac{(k + 1)(k + 2)}{2} \end{aligned} \tag{11}$$

■

The elegant result  $t_n = \frac{n(n+1)}{2}$  is also quite useful, and we will use it to help prove a few interesting identities that involve polygonal numbers.

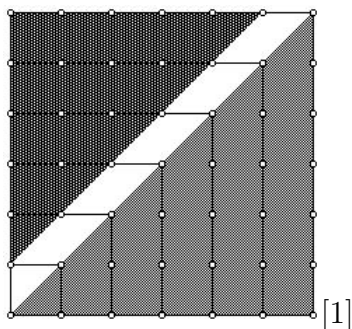
## 2.1 Interesting Identities

Consider the geometric representation of a square number. Notice that a long diagonal of points in the figure has the same number of points as the order of the square, since with each gnomon added both increase by one. Each diagonal of points next to the longest has one fewer point, and the number of points in subsequent diagonals decreases by one as the position moves farther from the longest diagonal. (The final “diagonal” has one point, the corners of the square not included in the long diagonal in consideration.)

We can see, then, that if we separate the geometric representation of a square number into two parts along this long diagonal – one part including the diagonal and all diagonals on one side, the other part including all the diagonals on the other side – that the sum of the points in these parts will equal triangular numbers. Specifically, they will be consecutive triangular numbers, the larger being of equal order to the square number.

Figure 3 illustrates this point for  $s_7$ , the seventh square number. The dark areas of the square denote the points that sum to the two triangular numbers; the lighter area signifies  $t_7$ , and the darkest area  $t_6$ .

Figure 3: Square Number as Sum of Two Triangular Numbers



We imagined this characteristic above for square numbers generally; that is, that a square number of any order is equal to the sum of consecutive triangular numbers, the larger being of the same order as the square. We will prove this theorem now.

**Theorem 2.3** *The sum of the consecutive triangular numbers,  $t_{n-1}$  and  $t_n$ , equals the square number  $s_n$ .*

**Proof:**

Let  $t_{n-1}$  and  $t_n$  be triangular numbers of orders  $n - 1$  and  $n$ , respectively. By the triangular

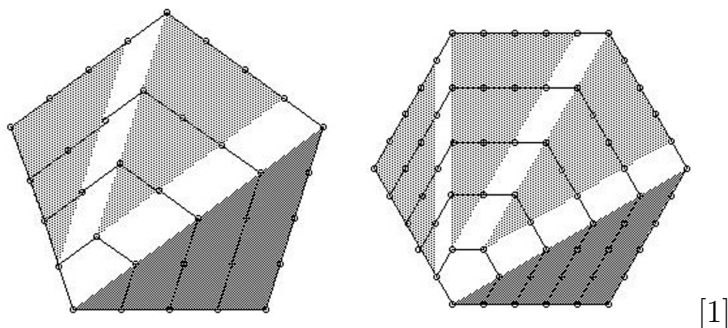
number generating formula, we know that  $t_{n-1} = \frac{(n-1)n}{2}$  and  $t_n = \frac{n(n+1)}{2}$ . Calculate

$$\begin{aligned} t_{n-1} + t_n &= \frac{(n-1)n}{2} + \frac{n(n+1)}{2} \\ &= \frac{(n-1)n + n(n+1)}{2} \\ &= \frac{2n^2}{2} \\ &= n^2 \end{aligned}$$

Because we know that  $n^2 = s_n$ , the proof is finished. [11] ■

This result is intriguing and requires some further attention. In particular, let us examine geometric representations of other polygonal numbers. Do similar characteristics hold? Figure 4 seems to suggest that they do. This figure displays  $p_5(5)$  and  $p_6(6)$ , the fifth pentagonal and sixth hexagonal numbers, respectively, showing how multiple triangular numbers constitute them. In particular, the pentagonal number is composed of  $t_5$  (the darker triangles) and two  $t_4$ s (the lighter triangles); the hexagonal number is composed of  $t_6$  and three  $t_5$ s.

Figure 4: Pentagonal and Hexagonal Numbers as Sums of Triangular Numbers



So the square number,  $p_4(n)$ , equaled  $t_n + t_{n-1}$ ; the pentagonal,  $p_5(n)$ , was equal to  $t_n + 2t_{n-1}$ ; and the hexagonal,  $p_6(n)$ , came to  $t_n + 3t_{n-1}$ . Is it possible that this pattern holds for polygonal numbers generally? That is, does  $p_d(n) = t_n + (d-3)t_{n-1}$ ? These figures suggest that that is the case, and the implication is significant. From this intuitive process we can produce a closed-form expression for the general polygonal number,  $p_d(n)$ , since we have proved the closed-form for triangular numbers. This expression we calculate as

$$\begin{aligned} p_d(n) &= t_n + (d-3)t_{n-1} \\ &= \frac{n(n+1)}{2} + (d-3)\frac{(n-1)n}{2} \\ &= \frac{(d-2)n^2 + (4-d)n}{2} \end{aligned}$$

Yet this intuitive expression is not proved for all  $d, n \in \mathbb{N}, d > 2$ . To achieve that, we look to the ideas of finite calculus.

### 3 Finite Calculus

Before we launch into the tenets of finite calculus, we should pause to examine the subject itself. As the name suggests, finite calculus is similar to conventional calculus. In fact, the two are analogous, yet while in calculus we needed to compute the area under a function, in finite calculus we want to compute the area under a sequence. Hence the name: sums in finite calculus are composed of a finite (rather than infinite, as in conventional calculus) set of terms. In other words, whereas calculus deals with real number functions, finite calculus works only on those of integers.

To illustrate the difference between finite and conventional calculus, consider how we use the notions of derivative, anti-derivative, and the Fundamental Theorem of Calculus to write the closed form solution of

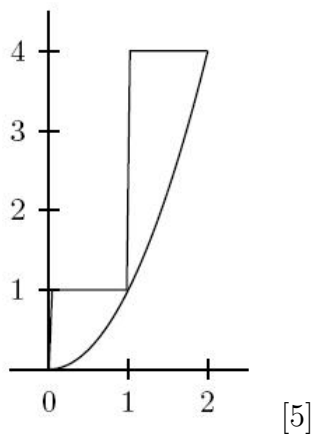
$$\int_a^b f(x)dx = F(b) - F(a),$$

where

$$\frac{d}{dx}F(x) = f(x).$$

Such a theorem for finite calculus would be helpful, as it would help us in our derivation of polygonal number generating formula. An exact replica of this theorem in the context of finite calculus, however, would be incorrect, as Figure 5 shows. The area under the two curves is obviously not equal.

Figure 5: Finite Summation Vs. Integration of  $n^2$  from 0 to 2





Before we can develop a similar theorem in the finite realm, however, we must address the basic idea of derivative in the context of finite sums. Recall from calculus that, for  $x, h \in \mathbb{R}$ ,

$$\frac{d}{dx}f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Since we are working with integers, the closest that  $h$  can be to 0 is 1. Thus, we arrive at a definition for the discrete derivative.

**Definition 3.1** *The “discrete derivative” of a function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is a new function,  $\Delta f : \mathbb{Z} \rightarrow \mathbb{Z}$ , defined as  $\Delta f(n) = f(n+1) - f(n)$ .*

To grasp this concept and how it compares to its conventional calculus analog we will consider a few examples. Remember from calculus that the derivative of a power looked like this:

$$\frac{d}{dx}x^m = mx^{m-1}.$$

Does this property hold for the discrete derivative? That is, does

$$\Delta n^m = mn^{m-1}$$

hold for all  $m$ ? Let’s check a few values of  $m$ . For  $m = 1$ ,

$$\Delta n = (n+1) - n = 1,$$

so the property holds in this case, but for  $m = 2$ ,

$$\Delta n^2 = (n+1)^2 - n^2 = 2n+1 \neq 2n.$$

Hence, the property does not hold for the discrete derivative. But observe: the failed discrete derivative was off by 1, and the discrete derivative of  $n$  equaled 1. Thus, we can discretely derive  $n^2 - n$  to equal  $2n$ , and, moreover, we can factor  $n^2 - n$  as  $n(n-1)$ . So,

$$\Delta(n^2 - n) = \Delta(n(n-1)) = 2n+1 - 1 = 2n.$$

So the discrete derivative of  $x \cdot (n-1)$ , rather than  $n \cdot n$  like in conventional calculus, gives us  $2n$ . This example suggests a new sort of power-property for discrete derivatives, one that involves products of factors that descend by 1. Such products are reminiscent of the factorial powers, e.g.  $3! = 3 \cdot 2 \cdot 1 = 6$ . Such a property for discrete derivatives does exist and does involve such powers, but first let us define them.

**Definition 3.2** *An integer  $n$  to a “factorial power”  $m$  equals  $n(n-1)(n-2)\dots(n-(m-1))$ , where  $m \in \mathbb{N}$ . Additionally, we give  $n^0 = 1$ . We write this expression as  $n^{\underline{m}}$ .*

These factorial powers will allow us to prove the property that we just asserted exists for discrete derivatives.

**Theorem 3.3** *For  $m \in \mathbb{N}$ ,  $\Delta(n^{\underline{m}}) = m \cdot n^{\underline{m-1}}$ .*

**Proof:**

Calculate

$$\begin{aligned}\Delta(n^m) &= (n+1)^m - n^m \\ &= ((n+1)n \cdots (n-(m-2))) \\ &\quad - (n(n-1) \cdots (n-(m-2))(n-(m-1))) \\ &= ((n+1) \cdot n^{m-1}) - (n^{m-1} \cdot (n-(m-1))) \\ &= ((n+1) - (n-(m-1))) \cdot n^{m-1} \\ &= m \cdot n^{m-1}. \quad [11]\end{aligned}$$

■

Now let us prove two useful theorems related to this property.

**Theorem 3.4** *The discrete derivative of the sum of functions is equal to the sum of the discrete derivative of the functions:*

$$\Delta(f(n) + g(n)) = \Delta f(n) + \Delta g(n)$$

**Proof:**

By definition,

$$\begin{aligned}\Delta(f(n) + g(n)) &= f(n+1) + g(n+1) - f(n) - g(n) \\ &= f(n+1) - f(n) + g(n+1) - g(n) \\ &= \Delta f(n) + \Delta g(n) \quad [5]\end{aligned}$$

■

**Theorem 3.5** *The discrete derivative of a constant times a function is equal to the constant times the discrete derivative of the function:*

$$\Delta(cf(n)) = c\Delta f(n)$$

**Proof:**

Similarly,

$$\begin{aligned}\Delta(cf(n)) &= cf(n+1) - cf(n) \\ &= c(f(n+1) - f(n)) \\ &= c\Delta f(n) \quad [5]\end{aligned}$$

■

Before we move on to prove any other theorems related to the discrete derivative, let us consider the finite analog to conventional calculus's anti-derivative.

**Definition 3.6** The “discrete anti-derivative” of a function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is a new function  $\Sigma f : \mathbb{Z} \rightarrow \mathbb{Z}$ , defined as  $\Sigma f(n) = F(n)$  such that  $\Delta F(n) = f(n)$ .

From the calculus analog, we can guess that the discrete anti-derivative is similar to

$$\int f(x)dx = F(x) + c,$$

where  $c$  is a constant. Indeed, we can prove the equivalent formula for the finite version, again using factorial powers.

**Theorem 3.7**

$$\Sigma n^m = \frac{n^{m+1}}{m+1} + C,$$

where  $C \in \mathbb{Z}, m \in \mathbb{N}, m \neq -1$ .

**Proof:**

Let  $C \in \mathbb{Z}, m \in \mathbb{N}, m \neq -1$ . Then

$$\begin{aligned} \Delta \left( \frac{n^{m+1}}{m+1} + C \right) &= \Delta \left( \frac{n^{m+1}}{m+1} \right) + \Delta(C) \\ &= \frac{1}{m+1} \Delta(n^{m+1}) + C - C \\ &= \frac{1}{m+1} (m+1)n^m \\ &= n^m \end{aligned}$$

■

We now can work toward a finite analog to the Fundamental Theorem of Calculus. From the calculus equivalent, we might expect it to look like this:

$$\sum_{n=a}^b f(n) = F(b) - F(a),$$

when  $\Sigma f(n) = F(n)$ , and  $n, a, b \in \mathbb{Z}$ . Yet a quick check of the sum of  $n$  from 1 to 5 shows this shows this conventional calculus-inspired formula to be incorrect:

$$\sum_1^5 n = 1 + 2 + 3 + 4 + 5 = 15,$$

yet

$$\left[ \frac{n^2}{2} \right]_1^5 = \frac{(5)(4)}{2} = 10.$$

We notice, however, that had we summed one fewer term on the top, the expression would have equalled the bottom. We can intuit that the actual theorem should have a left side summing to  $b - 1$  instead; we will prove that theorem now.

**Theorem 3.8 (Fundamental Theorem of Finite Calculus)** *If  $\Sigma f(n) = F(n)$ , then*

$$\sum_{n=a}^{b-1} f(n) = F(b) - F(a),$$

where  $a, b \in \mathbb{Z}$ .

**Proof:**

Let  $\Sigma f(n) = F(n)$ . This is equivalent to  $f(n) = \Delta F(n)$ . Calculate

$$\begin{aligned} \sum_{n=a}^{b-1} f(n) &= f(a) + f(a+1) + \dots + f(b-2) + f(b-1) \\ &= \Delta F(a) + \Delta F(a+1) + \dots + \Delta F(b-2) + \Delta F(b-1) \\ &= (F(a+1) - F(a)) + (F(a+2) - F(a+1)) + \dots + (F(b-1) - F(b-2)) + \\ &\hspace{15em} (F(b) - F(b-1)) \\ &= F(b) - F(a), \end{aligned}$$

because the telescoping sum collapses.

[11]

■

We now use the principles of finite calculus to procure a general formula for polygonal numbers and to continue to explore related subjects that show the interesting capabilities of finite calculus.

## 4 General Polygonal Number Generation

**Theorem 4.1** *A polygonal number with  $d$  sides and of order  $n$  is generated by the formula*

$$p_d(n) = \frac{(d-2)n^2 + (4-d)n}{2}.$$

**Proof:**

Let  $p_d(n)$  be a polygonal number with  $d$  sides and of order  $n$ . Then

$$\Delta p_d(n) = p_d(n+1) - p_d(n).$$

We know that  $p_d(n+1) - p_d(n)$  is just the gnomon that forms the polygonal number of order  $n+1$ . This gnomon,  $g_d(n+1)$ , we can express as  $n(d-2) + 1$ . Now, if we can take the discrete anti-derivative of  $g_d(n+1)$  and produce the function

$$\Sigma g_d(n+1) = G_d(n+1),$$

then we will be able to use the Fundamental Theorem of Finite Calculus to give us the summation formula for the gnomons. With the way we have defined polygonal numbers,

such a formula will be equivalent to a generating formula for the polygonal numbers themselves. In order to produce  $G_d(n+1)$  we must find an expression that, when mapped by the discrete derivative, gives  $n(d-2)+1$ . Following our earlier intuition, consider the triangular number  $t_n$ . We know that

$$t_n = \frac{n(n+1)}{2},$$

and thus

$$t_{n-1} = \frac{n(n-1)}{2},$$

which implies that

$$\Delta t_{n-1} = \frac{n(n+1)}{2} - \frac{n(n-1)}{2} = \frac{n^2 + n - (n^2 - n)}{2} = \frac{2n}{2} = n.$$

In short,  $\Delta t_{n-1} = n$ . Thus  $\Delta[t_{n-1}(d-2)] = n(d-2)$ , which gives us the first of our summands. The other, the constant 1, we need simply add another  $n$  to the expression to which we are taking the discrete derivative, since

$$\begin{aligned} \Delta(n) &= (n+1) - n \\ &= 1. \end{aligned}$$

As we have shown, then,

$$\Delta[t_{n-1}(d-2) + n] = n(d-2) + 1.$$

So calculate

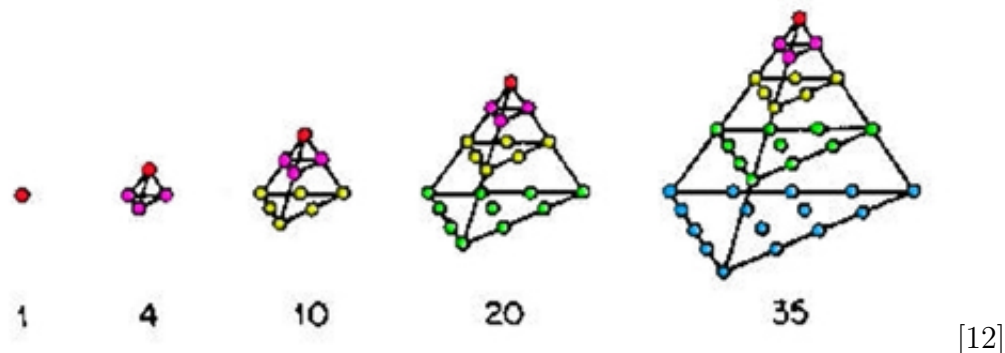
$$\begin{aligned} p_d(n) &= (d-2)t_{n-1} + n \\ &= (d-2) \left[ \frac{n(n-1)}{2} \right] + n \\ &= \frac{(d-2)n(n-1) + 2n}{2} \\ &= \frac{(d-2)n^2 + (4-d)n}{2}. \quad [9] \end{aligned}$$

■

## 5 Polyhedral Numbers

Polyhedral numbers are much like polygonal numbers, but the array of points that represent them are 3-dimensional, and the gnomons that generate them are themselves polygonal

Figure 6: The First Five Tetrahedral Numbers



numbers. We will consider the polyhedrals analogous to the polygonals we dealt with in the last section, triangular and square numbers. The corresponding polyhedral numbers are, respectively, tetrahedral and pyramidal numbers. Figure 5, which gives the graphical representation of the first five tetrahedral numbers, motivates the first definition well.

**Definition 5.1** A “tetrahedral number,”  $T_n$ , is the number of objects in the tetrahedral pyramid with  $n$  layers. The  $k^{\text{th}}$  layer of the pyramid is a triangle with  $t_k$  objects in it; so by definition,

$$T_n = t_1 + t_2 + \dots + t_{n-1} + t_n = \sum_{k=1}^n t_k.$$

As before, we can derive the formula that generates the tetrahedral numbers. This formula has special significance for our study of polygonal numbers. Notice that the  $n^{\text{th}}$  tetrahedral number is just the sum of the first  $n$  triangular numbers. So the formula that generates the  $n^{\text{th}}$  tetrahedral number will also be a summation formula for triangular numbers.

An intuitive method for deriving the formula for tetrahedral numbers does exist, though it is more complicated than in earlier cases. It is something of the two-dimensional analog of Gauss’s one-dimensional proof that  $t_n = \frac{n(n+1)}{2}$ . Imagine summing all the numbers in a triangle:

$$\begin{aligned} &1 \\ &1 + 2 \\ &1 + 2 + 3 \\ &1 + 2 + 3 + 4, \end{aligned}$$

where the  $k^{\text{th}}$  row is the triangular number  $t_k$  (doing so is the same as calculating the  $k^{\text{th}}$  tetrahedral number). We take three copies of the triangle, each one rotated by  $120^\circ$ , and add them.

$$\begin{array}{rcccc}
1 & + & 1 & + & 4 \\
1 + 2 & + & 2 + 1 & + & 3 + 3 \\
1 + 2 + 3 & + & 3 + 2 + 1 & + & 2 + 2 + 2 \\
1 + 2 + 3 + 4 & + & 4 + 3 + 2 + 1 & + & 1 + 1 + 1 + 1
\end{array}$$

The result in this case will be

$$\begin{array}{l}
6 \\
6 + 6 \\
6 + 6 + 6 \\
6 + 6 + 6 + 6,
\end{array}$$

or an array of terms that all equal  $k + 2$ . The number of these terms will equal  $t_k$ . Thus, the sum of the array will be their product, or  $t_k(k + 2)$ . We can say that this is just

$$\frac{n(n + 1)(n + 2)}{2}$$

by utilizing our triangular number formula. This expression, however, is equal to the sum of three copies of the  $k^{\text{th}}$  tetrahedral number, so one copy would be

$$\frac{n(n + 1)(n + 2)}{6}.$$

We can prove this intuitive result rigorously.

**Theorem 5.2** *The  $n^{\text{th}}$  tetrahedral number equals*

$$\frac{n(n + 1)(n + 2)}{6}.$$

**Proof:**

Let  $T_n$  be a tetrahedral number of order  $n$ . Then  $T_n = t_1 + t_2 + \dots + t_{n-1} + t_n = \sum_{k=1}^n t_k$ .

Proceed by induction on  $n$ . Check for  $n = 1$ :  $t_n = 1$  and

$$\frac{1(1 + 1)(1 + 2)}{6} = \frac{2(3)}{6} = 1,$$

so  $n = 1$  checks. Assume  $n = k$  holds true; that is,

$$T_k = \frac{k(k + 1)(k + 2)}{6}.$$

Prove for  $n = k + 1$ . Calculate

$$\begin{aligned}
 T_{k+1} &= t_1 + t_2 + \dots + t_{k-1} + t_k + t_{k+1} \\
 &= \frac{k(k+1)(k+2)}{6} + t_{k+1} \\
 &= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} \\
 &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{6} \\
 &= \frac{k^3 + 6k^2 + 11k + 6}{6} \\
 &= \frac{(k+1)(k+2)(k+3)}{6}. \tag{11}
 \end{aligned}$$

■

Let us now consider the polyhedral number that corresponds to the square numbers.

**Definition 5.3** A “pyramidal number,”  $P_n$ , is the number of objects in a pyramid with a square base and  $n$  layers, where the  $k^{\text{th}}$  layer of the pyramid is a square with  $s_k = k^2$  objects in it. So, by definition,

$$P_n = 1^2 + 2^2 + \dots + (n-1)^2 + n^2 = \sum_{k=1}^n k^2.$$

We can surmise a generating formula for this polyhedral number simply by noting a theorem we proved for the corresponding polygonal case:

$$n^2 = t_{n-1} + t_n.$$

So we might think that

$$\begin{aligned}
 P_n &= T_{n-1} + T_n \\
 &= \frac{n(n+1)(n-1)}{6} + \frac{n(n+1)(n+2)}{6} \\
 &= \frac{n(n+1)(2n+1)}{6}.
 \end{aligned}$$

Clearly, this guess makes sense geometrically. We prove the theorem now.

**Theorem 5.4** The  $n^{\text{th}}$  pyramidal number equals

$$\frac{n(n+1)(2n+1)}{6}.$$



**Proof:**

Let  $P_n$  be a pyramidal number of order  $n$ . Then  $P_n = \sum_{k=1}^n k^2$ . We can say that  $k^2 = t_{k-1} + t_k$ .

So

$$P_n = \sum_{k=1}^n (t_{k-1} + t_k) = \sum_{k=1}^n t_{k-1} + \sum_{k=1}^n t_k.$$

By definition of tetrahedral numbers we know then that  $P_n = T_{n-1} + T_n$ , and this must equal

$$\frac{(n-1)n(n+1)}{6} + \frac{n(n+1)(n+2)}{6} = \frac{n(n+1)(2n+1)}{6}. \quad [11]$$

■

Off the subject of polyhedral numbers but along the same lines as the last example (summing squares), what if we want to sum consecutive cubes? Can we find a closed-form expression for the sum of the  $n$  first consecutive powers of three? In fact, we can do so intuitively, but the method is not as obvious as some of those previous.

## 5.1 Consecutive Cube Summation

Nicomachus of Gerasa, yet another ancient Greek (circa 100 years before Diophantos), observed in his Introduction to Arithmetic the interesting pattern in sums of odd numbers:

$$\begin{aligned} 1 &= 1^3, \\ 3 + 5 &= 2^3, \\ 7 + 9 + 11 &= 3^3, \\ 13 + 15 + 17 + 19 &= 4^3, \end{aligned}$$

and so on. Such a patter would suggest that summing the cubes would be the same as summing consecutive odd numbers; e.g.,

$$\begin{aligned} 1 + 3 + 5 &= 1^3 + 2^3, \\ 1 + 3 + 5 + 7 + 9 + 11 &= 1^3 + 2^3 + 3^3 \\ 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 &= 1^3 + 2^3 + 3^3 + 4^3. \end{aligned}$$

So we can see that consecutive cube sums equal consecutive odd numbers. In general, though, how many consecutive odd numbers do we need? The above examples show the first two, three, and four cube sums needing 3, 6, and 10 consecutive odd numbers, respectively. Notice that 3 is the second triangular number, that 6 is the third, and that 10 is the fourth. So we can guess that the sum of the first  $n$  cubes will equal the first  $t_n$  consecutive odd numbers. We can express this idea as such:

$$1^3 + 2^3 + \dots + n^3 = 1 + 3 + \dots + (2t_n - 1).$$

But we have already worked with an object that is the sum of consecutive odd numbers – square numbers. So we can express the right side in a closed manner:

$$\begin{aligned} 1^3 + 2^3 + \dots + n^3 &= (t_n)^2 \\ &= \left(\frac{n(n+1)}{2}\right)^2 \\ &= \frac{n^2(n+1)^2}{4}. \end{aligned}$$

Our intuition (with Nicomachus’s help) has led us to a closed-form expression for the sum of consecutive cubes. We will prove it rigorously now.

**Theorem 5.5** *The sum of the first  $n$  consecutive cubes equals*

$$\frac{n^2(n+1)^2}{4}.$$

**Proof:**

Proceed by induction on  $1^3 + 2^3 + \dots + (n-1)^3 + n^3 = \sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$ .

Check for  $n = 1$ :

$$\sum_{k=1}^1 k^3 = 1^3 = 1$$

and

$$\frac{1^2(1+1)^2}{4} = \frac{2^2}{4} = 1,$$

so  $n = 1$  checks.

Assume that  $n = m$  holds true; that is,

$$1^3 + 2^3 + \dots + (m-1)^3 + m^3 = \frac{m^2(m+1)^2}{4}.$$

Prove for  $n = m + 1$ . Calculate

$$\begin{aligned} \sum_{k=1}^{m+1} k^3 &= 1^3 + 2^3 + \dots + m^3 + (m+1)^3 \\ &= \frac{m^2(m+1)^2}{4} + (m+1)^3 \\ &= \frac{m^4 + 6m^3 + 13m^2 + 12m + 4}{4} \\ &= \frac{(m+1)^2(m+2)^2}{4}. \end{aligned} \tag{11}$$

■

We have shown, thus, that we can find summation formulas for the powers up to three (power 1 was just the generation formula for triangular numbers!). Yet the method for finding the expression by  $n^3$  was becoming something of a stretch for our intuition. While there may be a way intuitively to derive a summation formula for  $n^4$ , it is not obvious, and we can imagine that finding the formula for considerably higher powers could be impossible. “Can we overcome the limits of our intuition and find formulas for these tricky sums?” we might ask ourselves. The answer, as you may have guessed, is yes! Doing so involves using the ideas from finite calculus.

## 6 General Power Summation

Recall that finite calculus works with factorial powers,  $n^m$  rather than regular powers. Let us examine the relationship between the two types of powers, which will help us put finite calculus to use in finding power-summation formulas.

$$\begin{aligned}n^0 &= n^0 \\n^1 &= n^1 \\n^2 &= n^2 + n^1\end{aligned}$$

These low powers are easy to convert, but for  $n^3$  the factorial power equivalent is not obvious. Following what seemed to be the emergent pattern, we can set  $n^3$  equal to

$$an^3 + bn^2 + cn^1$$

and solve for  $a$ ,  $b$ , and  $c$ . We find that we can do so successfully, yielding  $a = 1$ ,  $b = 3$ , and  $c = 1$ , so

$$n^3 = n^3 + 3n^2 + n^1$$

The pattern of coefficients strongly suggests that a special kind of number is involved in this process: the “Stirling numbers” (of the second type). Since not everyone is familiar with these numbers, we will define them formally.

**Definition 6.1** *The “Stirling numbers” are the numbers  $\left\{ \begin{matrix} m \\ k \end{matrix} \right\}$  such that*

$$\left\{ \begin{matrix} m \\ k \end{matrix} \right\} = k \cdot \left\{ \begin{matrix} m-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} m-1 \\ k-1 \end{matrix} \right\},$$

where  $m, k \in \mathbb{Z}$ . When  $k = 0$ :  $\left\{ \begin{matrix} m \\ k \end{matrix} \right\} = 0$  if  $m > 0$  and  $\left\{ \begin{matrix} m \\ k \end{matrix} \right\} = 1$  if  $m = 0$ . If  $k > m$  or  $k < 0$ , then  $\left\{ \begin{matrix} m \\ k \end{matrix} \right\} = 0$ .

Figure 7: The First Seven Rows of Stirling Numbers

				1			
			0	1			
		0	1	1			
	0	1	3	1			
	0	1	7	6	1		
	0	1	15	25	10	1	
0	1	31	90	65	15	1	[5]

Figure 1 shows the first seven rows of Stirlings. We can see that the coefficients in the expressions above match the top four rows.

This recognition leads us readily to a closed-form expression of the relationship between regular and factorial powers:

$$n^m = \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \cdot n^{\underline{k}},$$

where  $n, m, k \in \mathbb{Z}$ . Before we prove this theorem, however, we must prove another small theorem related to factorial powers.

**Theorem 6.2**

$$n^{\overline{m+k}} = n^m(n - m)^{\underline{k}}, \text{ where } n, m, k \in \mathbb{Z}.$$

**Proof:**

Let  $n, m, k \in \mathbb{Z}$ . Calculate

$$\begin{aligned} n^{\overline{m}(n - m)^{\underline{k}}} &= (n(n - 1) \cdots (n - (m + k - 1))) ((n - m)(n - m - 1) \cdots ((n - m) - (k - 1))) \\ &= n(n - 1) \cdots (n - (m - 1))(n - m)(n - (m + 1)) \cdots (n - (m + k - 1)) \\ &= n^{\overline{m+k}}. \end{aligned} \tag{11}$$

■

We now can prove our earlier theorem.

**Theorem 6.3** For  $n, m, k \in \mathbb{Z}$ ,

$$n^m = \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \cdot n^{\underline{k}}.$$

**Proof:**

Let  $n, m, k \in \mathbb{Z}$ . Proceed by induction on  $n^m = \sum_k \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \cdot n^k$ . (This makes sense to do

because the summands of  $n^m = \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \cdot n^k$  are just zero for  $k > m$  and  $k < 0$ .)

Check for  $m = 1$ :

$$\begin{aligned} n^1 &= n \\ \text{and} \\ \sum_k \left\{ \begin{matrix} 1 \\ k \end{matrix} \right\} \cdot n^k &= \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} \cdot n^0 + S(1, 1) \cdot n^1 \\ &= 0 \cdot 1 + 1 \cdot n \\ &= n, \end{aligned}$$

so  $n = 1$  checks.

Assume that  $m = j$  holds true; that is,

$$n^j = \sum_k \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \cdot n^k.$$

Prove for  $m = j + 1$ : Calculate

$$\begin{aligned} \sum_k \left\{ \begin{matrix} j+1 \\ k \end{matrix} \right\} \cdot n^k &= \sum_k \left[ k \cdot \left\{ \begin{matrix} j \\ k \end{matrix} \right\} + \left\{ \begin{matrix} j \\ k-1 \end{matrix} \right\} \right] \cdot n^k \\ &= \sum_k k \cdot \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \cdot n^k + \sum_k \left\{ \begin{matrix} j \\ k-1 \end{matrix} \right\} \cdot n^k \end{aligned}$$

Now, because we know that

$$\begin{aligned} n^{k+1} &= n^k(n-k)^1 \\ &= n \cdot n^k - k \cdot n^k \\ k \cdot n^k &= n \cdot n^k - n^{k+1}, \end{aligned}$$

we can say that

$$\begin{aligned} \sum_k \left\{ \begin{matrix} j+1 \\ k \end{matrix} \right\} \cdot n^k &= \sum_k \left\{ \begin{matrix} j \\ k \end{matrix} \right\} (n \cdot n^k - n^{k+1}) + \\ &\quad \sum_k \left\{ \begin{matrix} j \\ k-1 \end{matrix} \right\} \cdot n^k \\ &= \sum_k \left\{ \begin{matrix} j \\ k \end{matrix} \right\} (n \cdot n^k) - \sum_k \left\{ \begin{matrix} j \\ k \end{matrix} \right\} (n^{k+1}) + \sum_k \left\{ \begin{matrix} j \\ k-1 \end{matrix} \right\} \cdot n^k \end{aligned}$$

The last two summands actually cancel each other out because the factorial power is always one more than the lower parameter in the Stirling number. So,

$$\begin{aligned} \sum_k \left\{ \begin{matrix} j+1 \\ k \end{matrix} \right\} \cdot n^k &= n \cdot \sum_k \left\{ \begin{matrix} j \\ k \end{matrix} \right\} (n^{\underline{k}}) \\ &= n \cdot n^j \\ &= n^{j+1}. \end{aligned} \tag{11}$$

■

With this theorem in hand, we can use the ideas from finite calculus to produce a formula for any power sum we like. Let's try fifth powers.

**Theorem 6.4** *The sum of the  $n^{\text{th}}$  first consecutive fifth powers equals*

$$\frac{(2n^2 + 2n - 1)(n + 1)^2 n^2}{12}.$$

**Proof:**

Let  $n \in \mathbb{N}$ . We then can say that

$$n^5 = \sum_{k=0}^5 \left\{ \begin{matrix} 5 \\ k \end{matrix} \right\} \cdot n^{\underline{k}} = n^{\underline{1}} + 15n^{\underline{2}} + 25n^{\underline{3}} + 10n^{\underline{4}} + n^{\underline{5}}.$$

From Theorem 3.7 we know that

$$\Sigma n^5 = \frac{n^{\underline{2}}}{2} + 15 \frac{n^{\underline{3}}}{3} + 25 \frac{n^{\underline{4}}}{4} + 10 \frac{n^{\underline{5}}}{5} + \frac{n^{\underline{6}}}{6}.$$

The Fundamental Theorem of Finite Calculus thus implies that

$$\begin{aligned} \sum_{0 \leq k < n+1} k^5 &= \frac{(n+1)^{\underline{2}}}{2} + 15 \frac{(n+1)^{\underline{3}}}{3} + 25 \frac{(n+1)^{\underline{4}}}{4} + 10 \frac{(n+1)^{\underline{5}}}{5} + \frac{(n+1)^{\underline{6}}}{6} \\ &\quad - \left[ \frac{0^{\underline{2}}}{2} + 15 \frac{0^{\underline{3}}}{3} + 25 \frac{0^{\underline{4}}}{4} + 10 \frac{0^{\underline{5}}}{5} + \frac{0^{\underline{6}}}{6} \right] \\ &= \frac{(n+1)^{\underline{2}}}{2} + 5(n+1)^{\underline{3}} + \frac{25(n+1)^{\underline{4}}}{4} + 2(n+1)^{\underline{5}} + \frac{(n+1)^{\underline{6}}}{6} \\ &= \frac{2n^2 + 2n - 1)(n+1)^2 n^2}{12}. \end{aligned} \tag{11}$$

■

## 7 Conclusion

We have shown that, through a special type of calculus we can understand polygonal numbers, polyhedral numbers, and sums of powers more fully. It is worth noting that the process we took to find the finite sum of Theorem 6.4 can be replicated for any number of different kinds of sums that would otherwise be difficult if not impossible to solve. By applying to integers the tenets of calculus, powerful when used with real numbers, we can see their scope broaden – they become even more useful to us.

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