INTRODUCTION TO RAMSEY THEORY

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1. Introduction

In general, Ramsey theory deals with the guaranteed occurrence of specific structures in some part of a large arbitrary structure which has been partitioned into finitely many parts [1]. The field is named for Frank P. Ramsey who proved its first result, but many of its most significant contributions have come from Paul Erdős. Since then, the field has exploded. It now boasts many variations on classical Ramsey theory, and its effects are felt in widespread branches of mathematics. Here we will provide a brief introduction to graph theory in order to establish the necessary background needed to explore the basics of classical Ramsey theory. Many of the results we present served as the jumping-off points for entire new branches of the discipline, and this brief introduction is meant only to familiarize the reader with some key ideas and fundamental results.

2. Basics of Graph Theory

We should begin by first introducing some important concepts in graph theory that will allow us to develop Ramsey theory later. First, we will establish what a graph is and some important vocabulary used in the discussion of graphs.

Definition 2.1. A **graph** consists of two finite sets, V and E. Each element of V is called a **vertex**. The vertex set of a graph G is denoted by $V(G)$ or simply V. The elements of E are unordered pairs of vertices called **edges**. An edge connecting vertices u and v is denoted uv and u and v are said to be the edge's ends. The edge set is denoted by $E(G)$ or simply E. [2]

Instead of referring to a graph by it vertex and edge sets, more commonly we consider a visual depiction of a given graph. By convention, each element of the vertex set is represented by a dot, and each element of the edge set is represented by a line connecting two dots. Figure 1 illustrates two equivalent depictions of a given graph.

All graphs are generally divided into two main classes, simple graphs and multigraphs. The classifications depend the presence or absence of two features, loops and multiple edges.

Definition 2.2. An edge for which the two ends are the same is called a **loop** at the common vertex. A set of two or more edges of a graph is called a set of **multiple edges** if they have the same ends. [3]

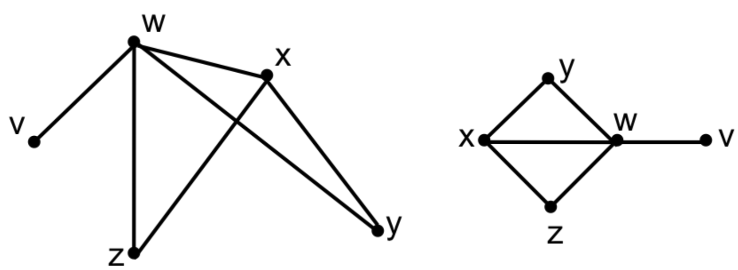

FIGURE 1. Two equivalent representations of a graph G where $V(G) = \{v, w, x, y, z\}$ and $E(G) = \{xy, yw, xz, wz, xw, wv\}$.

Definition 2.3. A graph is **simple** if it has no loops and no multiple edges. A graph is considered a **multigraph** if it contains at least one loop or multiple edge. [3]

We have already seen an example of a simple graph in Figure 1. Figure 2 provides an example of a multigraph. From here on, we will only be considering simple graphs in our discussions.

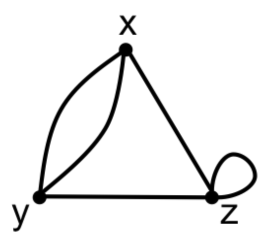

FIGURE 2. A multigraph. Note the multiple edges connecting vertices x and y and the loop connecting z to itself.

Next, we will define some important terms used when describing a graph.

Definition 2.4. The **order** of a graph G is the cardinality of $V(G)$, denoted $|V(G)|$. The **size** of a graph G is the cardinality of $E(G)$, denoted $|E(G)|$. [2]

Definition 2.5. Given $u, v \in V$, if $uv \in E$, then u and v are **adjacent**. If an edge e has a vertex v as an end, then v and e are **incident**. [2]

Definition 2.6. The **neighborhood** of a vertex v is the set of vertices adjacent to v. [2]

Definition 2.7. The **degree** of a vertex v, denoted $\deg(v)$, is the number of edges incident with v. [2]

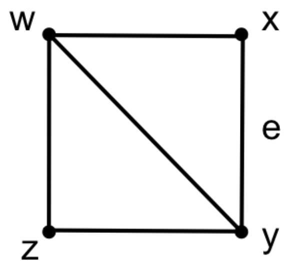

FIGURE 3. A graph G. Here, the order of G is 4 and the size of G is 5. w and y are adjacent, but x and z are not. x and edge e are incident. The neighborhood of w is the set $\{z, y, x\}$, and deg(w)=3.

Figure 3 provides an example to illustrate these concepts. Now we will present a simple, but useful result about graphs that we will use in proving more complicated results later.

Theorem 2.8. First Theorem of Graph Theory *In a graph G, the sum of the degrees of the vertices is equal to twice the number of edges. Alternatively, the number of vertices with odd degree is even.*

Proof. Let $S = \sum_{v \in V} \deg(v)$. Note that every edge is counted exactly twice in S because each edge is incident with two vertices. Thus $S = 2|E|$. Furthermore, it is clear that S is always even. Thus, we have

$$S = \sum_{v \in V} \deg(v) = \sum_{\substack{v \in V \text{such that} \\ \deg(v) \text{ is even}}} \deg(v) + \sum_{\substack{v \in V \text{such that} \\ \deg(v) \text{ is odd}}} \deg(v).$$

Summing the even degrees will result in an even number, so we know the first term of the above sum is even. Then, because an even plus an odd is odd, the only way that S can be even is if the second term of the above sum is also even. Because the second term is even and we know that the sum of an odd number of odd degrees is odd, there must be an even number of vertices with odd degree. [4]

$\square$

In our development of Ramsey theory, we will consider a special type of graph called a complete graph. Figure 4 provides several examples of complete graphs.

Definition 2.9. The **complete** graph on n vertices, K_n, is the graph of order n where $uv \in E$ for all $u, v \in V$. [3]

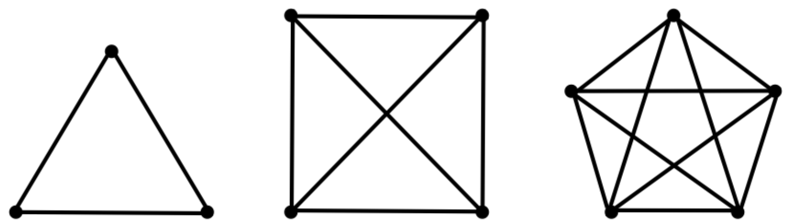

FIGURE 4. Examples of complete graphs, K_3 (left), K_4 (center), and K_5 (right).

3. Subgraphs

Sometimes there will be instances when we do not want to consider a whole graph. Instead, we will want to focus on only part of a given graph, or a subgraph.

Definition 3.1. A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say that $H \subseteq G$. [2]

By this definition, any graph G is considered a subgraph of itself. If we are interested in excluding this case, we would consider only proper subgraphs.

Definition 3.2. A subgraph H of a graph G is a **proper subgraph** of G if $V(H) \neq V(G)$ or $E(H) \neq E(G)$. [2]

Next, we will define two special types of subgraphs, induced subgraphs and cliques. Figures 5 and 6 provides examples of these special cases.

Definition 3.3. A subgraph H of G is an **induced subgraph** of G if each edge of G having its ends in $V(H)$ is also an edge of H. The induced subgraph of G with vertex set $S \subseteq V(G)$ is called the subgraph of G induced by S. [3]

Definition 3.4. A **clique** of G is a complete subgraph of G. The **clique number** of a graph G, denoted $\omega(G)$, is the order of the largest complete graph that is a subgraph of G. [3]

4. Colorings

One of the most important topics from graph theory to consider when discussing Ramsey theory is colorings. There are two main types of colorings, those on the vertices of a graph and those on the edges of a graph. Our discussion of Ramsey theory will require only edge colorings, but for the sake of completeness we will define both types.

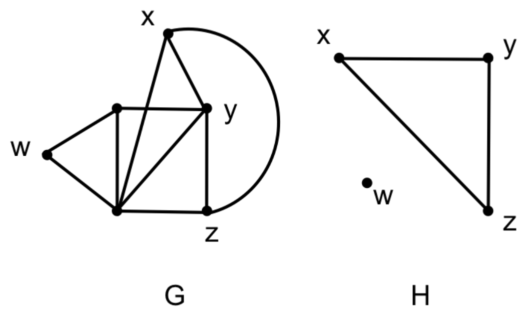

Figure 5. H is the subgraph of G induced by S where $S = \{w, x, y, z\}$.

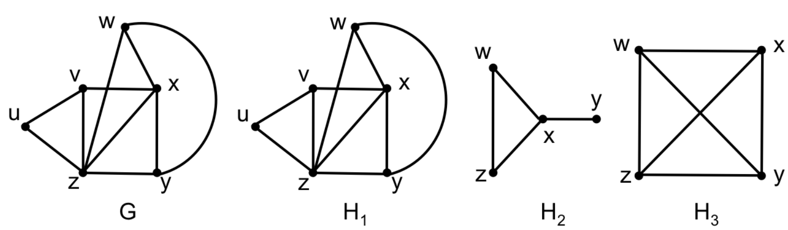

Figure 6. H_1, H_2, and H_3 are all subgraphs of G. Only H_2 and H_3 are proper subgraphs. H_3 is a clique. Because K_5 is not a subgraph of G, we can see that $\omega(G) = 4$.

Definition 4.1. Given a graph G, a **k-coloring of the vertices** of G is a partition of $V(G)$ into k sets $C_1, C_2, \ldots, C_k$ such that for all i, no pair of vertices from C_i are adjacent. If such a partition exists, G is said to be **k-colorable**. [4]

An example of a graph and a 4-coloring of its vertices is given in Figure 7.

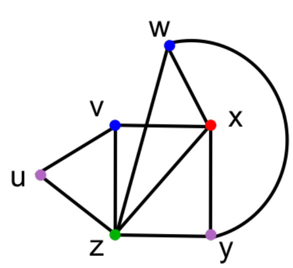

Figure 7. A graph G and a 4-coloring of its vertices such that $C_1 = \{u, y\}, C_2 = \{v, w\}, C_3 = \{x\}$, and $C_4 = \{z\}$.

Definition 4.2. Given a graph G, a **k-coloring of the edges** of G is any assignment of one of k colors to each of the edges of G. [4]

In our discussion of Ramsey theory, we will deal primarily with 2-colorings of the edges of graphs. By convention, the colors referred to are typically red and blue. An example of a graph and several 2-colorings of its edges is given in Figure 8.

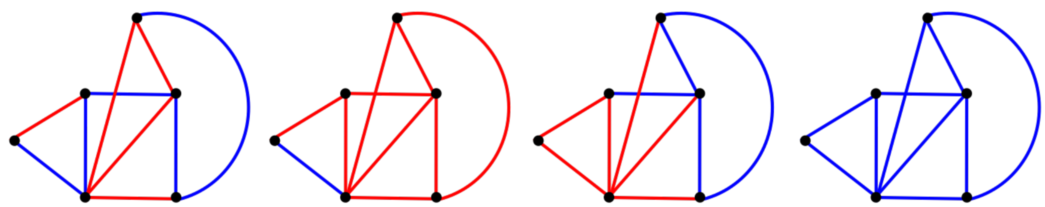

FIGURE 8. Four possible 2-colorings of the edges of a graph.

Definition 4.3. A subgraph H of G is **monochromatic** if all its edges receive the same color.

5. Ramsey Theory

Ramsey theory got its start and its name when Frank Ramsey published his paper "On a Problem of Formal Logic" in 1930 [5]. The theorem was proved in passing, as a means to a result about logic, but it turned out to be one of the first combinatorial results that widely attracted the attention of mathematicians. We will prove the theorem in terms of 2-colorings of the edges of complete graphs, but also discuss how the result can be viewed in terms of the old puzzle regarding mutual acquaintances and strangers at a party.

5.1. Ramsey's Theorem.

Theorem 5.1. Ramsey's Theorem *Given any positive integers p and q, there exists a smallest integer $n = R(p, q)$ such that every 2-coloring of the edges of K_n contains either a complete subgraph on p vertices, all of whose edges are in color 1, or a complete subgraph on q vertices, all of whose edges are in color 2.*

Proof. We will proceed by induction on $p + q$.

First we consider the base case in which $p + q = 2$. The only way this can be true is if $p = q = 1$, and it is clear that $R(1, 1) = 1$.

Now we assume that the theorem holds whenever $p + q < N$, for some positive integer N. Let P and Q be integers such that $P + Q = N$. Then $P + Q - 1 < N$, so by our assumption we know that $R(P - 1, Q)$ and $R(P, Q - 1)$ exist.

Consider any coloring of the edges of K_v in two colors c_1 and c_2, where $v \geq R(P-1,Q)+R(P,Q-1)$. Let x be a vertex of K_v. By the pigeonhole principle and because $v \geq R(P-1,Q)+R(P,Q-1)$, we know that of the $v-1$ edges that x is incident to, either $R(P-1,Q)$ edges are in color c_1 or $R(P,Q-1)$ edges are in color c_2.

If x is incident to $R(P-1,Q)$ edges of color c_1, consider the $K_{R(P-1,Q)}$ whose vertices are the vertices joined to x by edges of color c_1, that is the subgraph induced by the neighborhood of x. Because we know that $R(P-1,Q)$ exists, there are two possible cases to consider. One is that this graph contains a K_{P-1} with all edges in color c_1, in which case this K_{P-1} together with x forms a monochromatic K_P in color c_1. The other possibility is that $K_{R(P-1,Q)}$ contains a K_Q with all edges in color c_2. In either case, we can see that $R(P,Q)$ exists.

A parallel argument holds if x is incident to $R(P,Q-1)$ edges of color c_2, and K_v again contains one of the required monochromatic complete graphs.

Thus, $R(P,Q)$ exists, and in fact, because we chose v such that $v \geq R(P-1,Q)+R(P,Q-1)$, we know that $R(P,Q) \leq R(P,Q-1)+R(P-1,Q)$. [2]

□

If we consider this problem in terms of people at a party, Ramsey's Theorem guarantees that there is some smallest number of people at the party required to ensure that there is either a set of p mutual acquaintances or q mutual strangers. Thus, the old puzzle that asks us to prove that with any six people at a party, among them there must be a set of three mutual aquaintances or a set of three mutual strangers actually requires us to show that $R(3,3)=6$.

We should also note that Ramsey's Theorem can be generalized to account for colorings in any finite number of colors, not just 2-colorings.

Ramsey's Theorem guarantees that this smallest integer $R(p,q)$ exists but does little to help us determine what its value is, given some positive integers, p and q. In general, this is actually an exceedingly difficult problem.

5.2. **Ramsey Numbers.**

Definition 5.2. The integers $R(p,q)$ are known as classical **Ramsey numbers**.

Paul Erdős was a Hungarian mathematician who made huge contributions to the fields of combinatorics and graph theory. Commenting on the difficulty of determining Ramsey numbers, he said,

> "Suppose an evil alien would tell mankind 'Either you tell me [the value of R(5, 5)] or I will exterminate the human race.' ... It would be best in this case to try to compute it, both by mathematics and with a computer. If

> he would ask [for the value of R(6, 6)], the best thing would be to destroy him before he destroys us, because we couldn't. [6]"

Indeed, at the present relatively few nontrivial Ramsey numbers have been discovered. It follows from the definition of Ramsey's Theorem that for positive integers p and q, $R(p,q) = R(q,p)$, and we have already noted that $R(1,1) = 1$. Similarly, it is easy to see that for every positive integer k, $R(1,k) = 1$. Determining the values of $R(2,k)$ is only slightly more difficult.

Theorem 5.3. *$R(2,k) = k$ for all $k \geq 2$*

Proof. Choose some positive integer $k \geq 2$. First we will show that $R(2,k) > k-1$ by constructing a 2-coloring on K_{k-1} that contains neither a red K_2 nor a blue K_k. The coloring in which every edge is blue satisfies these requirements. It will certainly not contain a red K_2 and cannot possibly contain a blue K_k, so $R(2,k) > k-1$.

Next, suppose that the edges of K_k are 2-colored in some fashion. If any of the edges are red, then K_k will contain a red K_2. If none of the edges are red, then we are left with a blue K_k. So $R(2,k) \leq k$.

Thus, we can conclude that $R(2,k) = k$ for all $k \geq 2$.

□

The difficulty of determining additional Ramsey numbers grows quickly as p and q increase.

Theorem 5.4. *$R(3,3) = 6$.*

Proof. Consider any 2-coloring on K_6. Choose some vertex v from the graph. Because there are 5 edges incident to v, by the pigeon hole principle, at least three of these edges must be the same color. We will call them vx, vy, and vz, and we will suppose they are red. If at least one of xy, xz, or yz is red, then we have a red K_3. If none of these is red, then we have a blue K_3. Thus, $R(3,3) \leq 6$.

Next, consider the 2-coloring on K_5 as depicted in Figure 9. This coloring does not contain a monochromatic K_3 in either red or blue, so we know that $R(3,3) > 5$.

Thus, $R(3,3) = 6$. [4]

□

Theorem 5.5. *$R(3,4) = 9$.*

Proof. First, consider a coloring on K_8 as depicted in Figure 10. This coloring does not contain a red K_3 or a blue K_4, so we know that $R(3,4) > 8$.

Now we just need to show that $R(3,4) \leq 9$. Recall that we have already shown that $R(2,4) = 4$ and $R(3,3) = 6$.

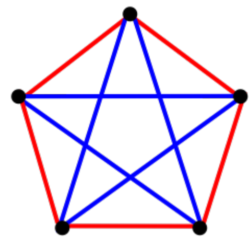

FIGURE 9. A 2-coloring on K_5 that contains no monochromatic K_3.

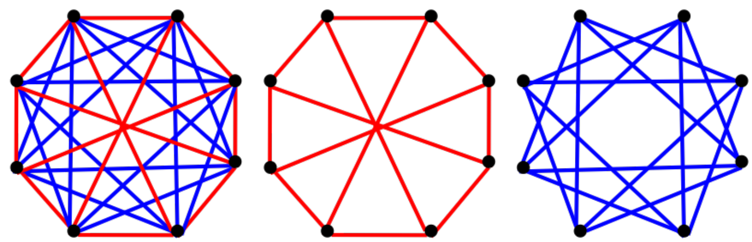

FIGURE 10. A 2-coloring on K_8 (left). Its red and blue monochromatic subgraphs are isolated to the right for closer inspection.

Let G be any complete graph of order at least 9. Consider any 2-coloring on G. Choose some vertex v from the graph. We will consider the following cases:

Case 1: Suppose that v is incident with at least four red edges. Let S be the set of vertices that are adjacent to v by a red edge. Because S contains at least four vertices and because $R(2,4) = 4$, the 2-coloring of the edges that are within the subgraph induced by S must produce either a red K_2 or a blue K_4 within this subgraph itself. If there is a red K_2 within the subgraph, then, together with v, we have found a red K_3 in G. If there is a blue K_4 within the subgraph induced by S then we obviously have already found one within G.

Case 2: Suppose that v is incident with at least six blue edges. Let T be the set of vertices that are adjacent to v by a blue edge. Because T contains at least six vertices and because $R(3,3) = 6$, the 2-coloring of the edges that are within the subgraph induced by T must produce either a red K_3 or a blue K_3 within the subgraph itself. If there is a red K_3 within this subgraph, we obviously also have one within G. If there is a blue K_3 within the subgraph, then, together with v, we have found a blue K_4 within G.

Case 3: Suppose that v is incident with fewer than four red edges and fewer than six blue edges. Then the order of G must be at most 9. However, we already assumed that the order of G is at least 9, so the order of G is exactly 9. Then, we know that v is incident with exactly three red edges and exactly five blue edges. Because v was chosen arbitrarily, we can assume this holds true for all nine vertices of G. If we consider the red monochromatic subgraph of G, we have a graph with nine vertices, each of which has degree 3. However, this provides a contradiction because Theorem 2.8 states that the number of vertices with odd degree must be even. Thus, this case can never occur.

We conclude that $R(3,4) = 9$. [4]

□

Clearly, determining the value of even very small Ramsey numbers is not a trivial problem. In fact, the only other known classical Ramsey numbers are the following: $R(3,5) = 14$, $R(3,6) = 18$, $R(3,7) = 23$, $R(3,8) = 28$, $R(3,9) = 36$, $R(4,4) = 18$, and $R(4,5) = 25$. There is only one nontrivial Ramsey number known for the generalized version of the theorem that allows more than two colors, $R(3,3,3) = 17$.

In cases for which it is not yet possible to determine the exactly value of a Ramsey number, it is often still possible to establish bounds for the value.

5.3. **Bounds on Ramsey Numbers.**

As we have already seen, in many cases we are able to establish a lower bound for some Ramsey number $R(p,q)$ by finding a complete graph K_n and a 2-coloring of K_n such that there is no monochromatic K_p or K_q. In this case we would know that $R(p,q) > n$.

There has been some progress with determining bounds beyond simply constructing a 2-coloring. Some of the most important results are given below. The first of these we have actually already proven when we originally proved Theorem 5.1.

Theorem 5.6. *If $p \geq 2$ and $q \geq 2$, then $R(p,q) \leq R(p-1,q) + R(p,q-1)$.*

The next result was originally proved by Erdős and Szekeres in 1935 [7]. It involves the binomial coefficient $\binom{n}{k}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $0 \leq k \leq n$. Our proof relies on Pascal's rule which we will prove first.

Lemma 5.7. Pascal's Rule $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

Proof.

$$
\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-(k+1))!} \\
&= \frac{n!(k+1)}{k!(n-k)!(k+1)} + \frac{n!(n-k)}{(k+1)!(n-(k+1))!(n-k)} \\
&= \frac{n!(k+1)}{(n-k)!(k+1)!} + \frac{n!(n-k)}{(k+1)!(n-k)!} \\
&= \frac{n!(k+1) + n!(n-k)}{(k+1)!(n-k)!} \\
&= \frac{n!(k+1+n-k)}{(k+1)!(n-k)!} \\
&= \frac{n!(n+1)}{(k+1)!(n-k)!} \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} \\
&= \frac{(n+1)!}{(k+1)!((n+1)-(k+1))!} \\
&= \binom{n+1}{k+1}.
\end{aligned}
$$

□

Theorem 5.8. *For integers p and q such that $p, q \geq 2$, $R(p,q) \leq \binom{p+q-2}{p-1} = \frac{(p+q-2)!}{(p-1)!(q-1)!}$.*

Proof. We will proceed by double induction on p and q.

First we consider the base case in which $p = q = 2$. We have already seen that $R(2,2) = 2$, so it is easy to find that $R(2,2) = 2 \leq \frac{(2+2-2)!}{(2-1)!(2-1)!} = \frac{2!}{(1!)(1!)} = 2$. Clearly the theorem holds in this case.

Now we assume that the theorem holds for $R(p-1,q)$ and $R(p,q-1)$. By Theorem 5.6, we know that $R(p,q) \leq R(p-1,q) + R(p,q-1)$. So with our assumption we have

$$
\begin{aligned}
R(p,q) &\leq R(p-1,q) + R(p,q-1) \\
&\leq \binom{(p-1)+q-2}{(p-1)-1} + \binom{p+(q-1)-2}{p-1} \\
&= \binom{p+q-3}{p-2} + \binom{p+q-3}{p-1}.
\end{aligned}
$$

Finally, by Lemma 5.7, we know that

$$\binom{p+q-3}{p-2} + \binom{p+q-3}{p-1} = \binom{p+q-2}{p-1}.$$

Thus, we can conclude that

$$R(p,q) \leq \binom{p+q-2}{p-1}.$$

[8] □

Finally, we will provide a lower bound for a special class of Ramsey numbers for which $p = q$.

Theorem 5.9. *If n is an integer satisfying $\binom{n}{a} < 2^{\binom{a}{2}-1}$, then $R(a,a) > n$.*

Proof. Let a be a positive integer, and let n be an integer such that $\binom{n}{a} < 2^{\binom{a}{2}-1}$.

To determine that $R(a,a) > n$, we need to show that there is a 2-coloring on K_n that contains neither a blue monochromatic K_a nor a red monochromatic K_a. We will proceed by counting the colorings that do not meet this criterion, that is, colorings of K_n that contain either a blue monochromatic K_a or a red monochromatic K_a.

Let S be any subset containing a vertices of K_n. Let A_S be the set of 2-colorings of K_n in which the subgraph induced by S is either all blue or all red. We can count the number of colorings in A_S by considering that there are two ways to color the edges in the subgraph induced by S so that it is monochromatic (either all blue or all red) and there are $2^{\binom{n}{2}-\binom{a}{2}}$ ways to color the remaining $\binom{n}{2} - \binom{a}{2}$ edges of K_n. So, $|A_S| = 2 \cdot 2^{\binom{n}{2}-\binom{a}{2}}$.

We see that the number of 2-colorings containing either a red K_a or a blue K_a is given by the cardinality of the union of all the A_S's for all possible subsets S containing a vertices,

$$\left| \bigcup_{S:|S|=a} A_S \right|.$$

We know that the union can be no bigger than the sum of the sizes of the sets, so

$$\left| \bigcup_{S:|S|=a} A_S \right| \leq \sum_{S:|S|=a} |A_S|.$$

We have already determined that $|A_S| = 2 \cdot 2^{\binom{n}{2}-\binom{a}{2}}$, so

$$
\begin{aligned}
\left|\bigcup_{S:|S|=a} A_S\right| &\leq \sum_{S:|S|=a} |A_S| \\
&= \sum_{S:|S|=a} 2 \cdot 2^{\binom{n}{2}-\binom{a}{2}} \\
&= \binom{n}{a} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{a}{2}}.
\end{aligned}
$$

So far we have shown that the number of 2-colorings containing a blue K_a or a red K_a is less than or equal to $\binom{n}{a} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{a}{2}}$. Therefore, if $\binom{n}{a} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{a}{2}}$ is less than the total number of possible 2-colorings of K_n, there must be a coloring containing neither a red K_a nor a blue K_a.

There are $2^{\binom{n}{2}}$ possible colorings of K_n, and we have assumed that $\binom{n}{a} < 2^{\binom{a}{2}-1}$. Thus,

$$
\begin{aligned}
\binom{n}{a} &< 2^{\binom{a}{2}-1} \\
\binom{n}{a} \cdot 2 &< 2 \cdot 2^{\binom{a}{2}-1} \\
\binom{n}{a} \cdot 2 &< 2^{\binom{a}{2}} \\
\binom{n}{a} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{a}{2}} &< 2^{\binom{a}{2}} \cdot 2^{\binom{n}{2}-\binom{a}{2}} \\
&= 2^{\binom{a}{2}+\binom{n}{2}-\binom{a}{2}} \\
&= 2^{\binom{n}{2}}.
\end{aligned}
$$

Hence, there must be a 2-coloring of K_n containing neither a red K_a nor a blue K_a, and we can conclude that $R(a, a) > n$. [8]

$\square$

In addition to the bounds given by these theorems, a number of specific bounds have also been found. Table 1 summarizes all the Ramsey numbers and bounds known at the present for $p, q \leq 10$ [9].

Table 1. Known Ramsey numbers $R(p, q)$ and bounds.

p, q	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	6	9	14	18	23	28	36	40-43
4	1	4	9	18	25	35-41	49-61	56-84	73-115	92-149
5	1	5	14	25	43-49	58-87	80-143	101-216	125-316	143-442
6	1	6	18	35-41	58-87	102-165	113-298	127-495	169-780	179-1171
7	1	7	23	49-61	80-143	113-298	205-540	216-1031	233-1713	298-2826
8	1	8	28	56-84	101-216	127-495	216-1031	282-1870	317-3583	317-6090
9	1	9	36	73-115	125-316	169-780	233-1713	317-3583	565-6588	580-12677
10	1	10	40-43	92-149	143-442	179-1171	289-2826	317-6090	580-12677	798-23556

6. Open Problems

Clearly this field still offers a huge number of open problems. The most obvious of which are finding more Ramsey numbers and improving the bounds we currently know. However, there are many related open problems that go beyond this. We will highlight a few of these problems, and the reader should note that, in most cases, there is a cash reward available for the solutions.

Proposition 6.1. *The limit* $\lim_{n\to\infty} R(n,n)^{\frac{1}{n}}$ *exists. [1]*

Problem 6.2. *Determine the value of* $c := \lim_{n\to\infty} R(n,n)^{\frac{1}{n}}$*. [1]*

Problem 6.3. *Prove or disprove that* $R(4,n) > \frac{n^3}{log^c n}$ *for some* c*, provided* n *is sufficiently large. [1]*

Proposition 6.4. *For fixed* k*,* $R(k,n) > \frac{n^{k-1}}{log^c k}$ *for a suitable constant* $c > 0$ *and* n *large. [1]*

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